Brief paper

$\mathcal{H}_\infty$ filtering for rectangular discrete-time descriptor systems

João Y. Ishihara a, Marco H. Terra b,*, B.M. Espinoza a

a Electrical Engineering Department, Faculty of Technology, University of Brasília, DF, Brazil
b Electrical Engineering Department, University of São Paulo at São Carlos C.P.359, São Carlos, SP, 13566-590, Brazil

A R T I C L E   I N F O

Article history:
Received 3 January 2008
Received in revised form 13 February 2009
Accepted 10 March 2009
Available online 16 April 2009

Keywords:
Descriptor systems
$\mathcal{H}_\infty$ filtering
Estimation

A B S T R A C T

This paper deals with the $\mathcal{H}_\infty$ recursive estimation problem for general rectangular time-variant descriptor systems in discrete time. Riccati-equation based recursions for filtered and predicted estimates are developed based on a data fitting approach and game theory. In this approach, the nature determines a state sequence seeking to maximize the estimation cost, whereas the estimator tries to find an estimate that brings the estimation cost to a minimum. A solution exists for a specified $\gamma$-level if the resulting cost is positive. In order to present some computational alternatives to the $\mathcal{H}_\infty$ filters developed, they are rewritten in information form along with the respective array algorithms.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

The study of estimation and control of descriptor systems (also known as singular systems or implicit systems) is motivated by the fact that systems in descriptor formulation frequently arises naturally in economical systems (Luenberger, 1977), image modeling (Hasan & Azim-Sadjani, 1995), and robotics (Mills & Goldenberg, 1989). Most commonly, filtering problems for discrete-time descriptor systems are featured in order to minimize the estimate error variance. In this case, one can find elegant recursive solutions for the so-called generalized Kalman filters (see for instance Darouach, Zasadzinski, and Mehdi (1993), Deng and Liu (1999), Keller, Nowakowski, and Darouach (1992), Nikoukhah, Campbell, and Delebecque (1999), Terra, Ishihara, and Padoan (2007a), Terra, Ishihara, and Padoan (2007b) and Zhang, Xie, and Soh (1999)). The approaches developed in these references are based on least square method, maximum likelihood criterion, minimum-variance estimation, and ARMA innovation model.

More recently, some attention has been focused on $\mathcal{H}_\infty$ filtering for descriptor systems. In Xu and Lam (2006) and Xu, Lam, and Zou (2003), are developed estimates for regular descriptor systems considering this class of filter based on linear matrix inequalities (LMIs). In Wang, Dai, and Liu (2006), a fixed-lag smoothing is proposed for regular descriptor systems in discrete-time. However, to the best of the authors’ knowledge, recursive $\mathcal{H}_\infty$ predicted and filtered estimates for rectangular discrete-time descriptor systems have not appeared in the literature yet. Rectangular descriptor systems are more general than square systems. They can admit, for example, multiple solutions.

As it is well known, the Kalman filter is a recursive algorithm that requires the propagation of error covariance matrices in terms of a Riccati equation. Historically, soon after its introduction by Kalman, it was realized that the original algorithm should be modified in order to increase the speed, minimize the cost, and decrease the computational complexities. Information filtering has been considered as one of these modifications, it provides an alternative approach to the covariance recursions. The filter in information form computes the inverse of the covariance matrix (information matrix), $P^{-1}$. Applications where this approach is more appropriate include multi-sensor environments (Kim & Hong, 2005) and certain classes of large interconnected systems (Bierman, 1977). The distinct forms of implementations of the Kalman filter have been also accomplished by the square-root array algorithms (Dyer & McReynolds, 1969; Harter, 1972; Kaminski, Bryson, & Schmidt, 1971). These algorithms provide better numerical conditioning and a reduced dynamical range of the filter parameters, which lead to more stable algorithms (Hassibi, Kailath, & Sayed, 2000).

In general, for $\mathcal{H}_\infty$ estimates of standard state-space systems, closed-loop transfer functions from the unknown disturbances to the filtered and predicted errors are designed in order to satisfy a prescribed $\mathcal{H}_\infty$-norm constraint. In this paper, the discrete-time $\mathcal{H}_\infty$ filter problem is solved in its full generality where all
matrices of the descriptor system are allowed to be rectangular. In this case a closed-loop transfer function may not exist, and the existence of the a priori filter does not assure the existence of the a priori filter. Therefore, classical state-space arguments based on innovation cannot be used to deduce this kind of filter. The discrete-time $\mathcal{H}_\infty$ filters developed in this paper are based on data fitting arguments combined with two-players game theory. The first player, the nature, determines a state sequence seeking to maximize the estimation cost, whereas the estimator tries to find an estimate that brings the quadratic cost to a minimum. A solution exists for a specified $\gamma$-level if the resulting cost is positive.

This paper is organized as follows. In Section 2, the $\mathcal{H}_\infty$ descriptor data fitting problems are defined. In the Sections 2.1 and 2.2 the recursive $\mathcal{H}_\infty$ singular filtered and predicted estimates are presented. In Section 3, the information form for the $\mathcal{H}_\infty$ filtered and predicted estimates are deduced and in Section 4 square-root array algorithms for the information filters are developed.

2. $\mathcal{H}_\infty$ descriptor data fitting problem

The $\mathcal{H}_\infty$ descriptor data fitting problem we are dealing with in this paper is stated as follows. Suppose given a sequence of sets of observations

\[ \{y_0\}, \{y_0, y_1\}, \ldots, \{y_0, \ldots, y_k\}, \ldots \]  

and a system of linear equations of the form

\[ E_{i+k}x_{i+k} = F_i x_i + u_i \]

\[ y_i = H_i x_i + v_i, \quad i = 0, 1, 2, \ldots \]

where $E_i$, $F_i$, $H_i$, and $L_i$ are real rectangular matrices of appropriate dimensions. For each stage (time instant or position) $k$, we can state two $\mathcal{H}_\infty$-type data fitting problems. The a posteriori $\mathcal{H}_\infty$ data fitting problem is to find sequences

\[ \{x_0, \ldots, x_k\}, \{u_0, \ldots, u_k\}, \{v_0, \ldots, v_k\}, \{z_0, \ldots, z_k\}, \]

\[ \{w_0, \ldots, w_k\} \]

which satisfy (2)-(4) and

\[ \min_{\{x_i\}^k_{i=0}, \{z_i\}^k_{i=0}, \{w_i\}^k_{i=0}} \left( \sum_{i=0}^{k} \|y_i - H_i x_i\|^2 + \sum_{i=0}^{k} \|x_i - F_i x_i + u_i\|^2 \right) > 0 \]

\[ \min_{\{z_i\}^k_{i=0}} \left( \sum_{i=0}^{k} \|z_i - L_i x_i\|^2 \right) > 0 \]

for $k = -1$ and

\[ J_k^f := \|E_0 x_{0}\|^2 + \sum_{i=0}^{k} \|y_i - H_i x_i\|^2 + \sum_{i=0}^{k} \|x_i - F_i x_i + u_i\|^2 \]

\[ + \sum_{i=0}^{k} \|x_i - F_i x_i + u_i\|^2 - y^2 \sum_{i=0}^{k} \|z_i - L_i x_i\|^2 \]

\[ \quad \|z_{0} - L_0 x_0\|^2 < \gamma \]

for $k \geq 0$. In (6)-(9), the matrices $E_0$ and $F_1$ are supposed given and $x_0$ denotes an initial guess for $x_0$. Usually, we have $E_0 = F_1 = 0$. The notation $\|x\|^2$ is used to represent $x'V_k x$. The weighting matrices $S_i > 0$, $V_i > 0$ and $W_i > 0$ are supposed given and of appropriate dimensions.

Therefore, in this paper we aim to create a sequence of direct resolutions for the $k$-horizon data fitting problem for which the observed data are updated. In the next section, it will be shown that the $(k+1)$-horizon data fitting solution can be written in terms of the $k$-horizon solution.

At this point, it is important to make some additional comments.

Remark 1. The general descriptor formulation considered in this paper may be useful, for example, in image processing and economic systems where rectangular models are common. In image processing, the index $i$ in (2) is usually related to space rather than time. Therefore, in this case, the standard state-space causal dynamics interpretation can be restrictive and even not useful.

Remark 2. There are some particular issues in relation to the rectangular descriptor systems we are considering in this paper which are important to emphasize.

- Usually regularity and causality are assumed in order to enable $u_k$ to be a free input sequence and to assure that $\{u_i\}$ and the initial state $x_0$ lead to a unique solution $\{x_i\}$. In this paper, the system (2) may not be regular. For an admissible input sequence and initial state, it is possible to have more than one solution. However, it can be seen that, as defined, if the $k$-horizon data fitting problem can be solved then it provides a particular admissible sequence $\{u_i\}$ and initial state $x_0$.

- A transfer function between the input given by the dynamic and measurement fitting errors $\{u_i\}$ and $\{v_i\}$, respectively, and the output fitting error $\{u_i\}$ may not exist since the causality is not assured. However, even if a causal relation does not exist, the solution of (5) or (8) assures that a gain relation between these fitting errors is always less than $\gamma^2$. In fact, considering only $k = 0$ for easy presentation, we can relate (5) to the gain

\[ \|z_{0} - L_0 x_0\|^2 \|z_{0} - L_0 x_0\|^2 \|y_0 - H_0 x_0\|^2 \|y_0 - H_0 x_0\|^2 < \gamma^2 \]

This relation shows that the $\mathcal{H}_\infty$ data fitting provides a particular trade-off between the fitting errors which can be tuned by the parameter $\gamma$.

Remark 3. When we consider $E_j = I$ for all $j$ (the classical state-space case), we can compare the proposed functionals to those provided in Hassibi, Sayed, and Kailath (1999) and Theodor and Shaked (1994). In (7), at each instant $k$, the errors of the estimates $z_{ki}$ for $i = 0, \ldots, k$ are weighted. In our approach, after solving the problem for $k$ and $k + 1$, we deduce the recursive relation between $z_{k+1}$ and $z_{k+1+k}$. In Hassibi et al. (1999, p. 84) and Theodor and Shaked (1994), the previously filtered estimates $z_{il}$, $i = 0, \ldots, k$ are directly weighted. The result decreasing $z_{i+1}$ and $z_{i+1}$ are then related for $i = 0, \ldots, k$. Similarly, the functional (9) weights the fitting errors $z_{ki} - L_k x_{ki}$ for $i = 0, \ldots, k + 1$, while the reference Hassibi et al. (1999) considers directly the prediction errors $z_{i+1} - L_{i+1} x_{i+1}$ for $i = 0, \ldots, k$. Besides interpretation, the indexes for the filter
functionals are in fact different. In the state-space context this difference is not relevant. However, for descriptor systems if we change in (2) the index limit from \( k - 1 \) to \( k \), the problem is different and, in consequence, the existence condition of the filter turns to be different. It will be shown in this paper that with the functionals we have adopted for descriptor systems, the resulting expressions for state-space systems have more resemblance to those standard \( H_\infty \) filters we have found in the literature.

### 2.1. \( H_\infty \) descriptor filter

The game problem (5)–(7) states that, for each \( k \geq 0 \), given the measurements \( \{ y_i \}_{i=0}^{\infty} \), the first player produces the estimates \( \hat{z}_i \) of \( z_i \), \( i = 0, 1, \ldots, k \). The second player, the nature, chooses the sequence of estimates \( \{ \hat{z}_i \}_{i=0}^{\infty} \) attempting to disrupt the first player estimate accuracy. Now, in order to solve the \( H_\infty \) filtering problem (5)–(7), observe that it is not necessary to maximize \( j_k^f \) over \( \{ z_i \}_{i=0}^{\infty} \). The \( H_\infty \) filter exists at the instant \( k \), if and only if, there exists a \( \{ \hat{z}_i \}_{i=0}^{\infty} \) such that \( j_k^f (\{ y_i \}_{i=0}^{\infty}, \{ \hat{z}_i \}_{i=0}^{\infty}, \{ x_i \}_{i=0}^{\infty}) \) has a minimum \( \{ \hat{z}_i \}_{i=0}^{\infty} \) for which \( j_k^f (\{ y_i \}_{i=0}^{\infty}, \{ \hat{z}_i \}_{i=0}^{\infty}, \{ x_i \}_{i=0}^{\infty}) > 0 \). Therefore, let us first consider the minimization problem in (5). For each \( k \geq 0 \), it is easy to show that (7) can be rewritten as

\[
J_k = (x_k x_{ik} - \beta_k)^T R_k (x_k x_{ik} - \beta_k)
\]

where

\[
x_{ik} := \begin{bmatrix} x_{ik} \\ \vdots \\ x_{0k} \end{bmatrix}, \quad \beta_k = \begin{bmatrix} z_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad R_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

\[
\tilde{A}_k = \begin{bmatrix} \tilde{A}_k & 0 & 0 & 0 \\ 0 & \tilde{A}_{k-1} & 0 & 0 \\ 0 & 0 & \tilde{A}_1 & 0 \\ 0 & 0 & 0 & \tilde{A}_0 \end{bmatrix}, \quad \tilde{E}_k = \begin{bmatrix} \tilde{E}_k \\ \tilde{H}_k \end{bmatrix},
\]

\[
R_i := \begin{bmatrix} \tilde{W}_{i-1} & 0 & 0 & 0 \\ 0 & \tilde{V}_i & 0 & 0 \end{bmatrix}, \quad A_{i-1} = \begin{bmatrix} -F_{i-1} \\ 0 \\ \cdots \\ 0 \end{bmatrix}.
\]

\[
Z_i := \begin{bmatrix} 0 \\ y_i \\ \vdots \\ z_{i0} \end{bmatrix}, \quad 1 \leq i \leq k, \quad Z_0 := \begin{bmatrix} f_{x0} \\ y_0 \\ z_{00} \end{bmatrix}.
\]

\[
W_{-1} := P_0.
\]

Considering the above variables, it is easy to see that for all \( k \geq 0 \) we have the following recurrent relations

\[
\begin{align*}
\tilde{R}_k &= \begin{bmatrix} 0 & 0 \\ 0 & \tilde{R}_{k-1} \end{bmatrix}, \\
\tilde{A}_k &= \begin{bmatrix} 0 \\ \tilde{x}_k \\ \vdots \\ \tilde{x}_{0k} \end{bmatrix}, \\
\tilde{A}_{k-1} &= \begin{bmatrix} A_{k-1} \\ 0 \\ \cdots \\ 0 \end{bmatrix}.
\end{align*}
\]

The next lemma will be useful to deduce the \( H_\infty \) descriptor estimators.

**Lemma 4** (Clements & Anderson, 1978). Consider matrices \( \alpha \) and \( R \) and column vectors \( \beta \) and \( x \) of appropriate dimensions with \( R \) symmetric. For any \( \beta \) we have

\[
\inf_{\alpha \in \mathbb{R}} (\alpha \beta - \beta)^T R (\alpha \beta - \beta) = -\infty
\]

if and only if \( \alpha^T R \alpha \geq 0 \) and \( \text{Ker}(\alpha^T R \alpha) \subset \text{Ker}(R \alpha) \). If the minimum is attained, it is unique if and only if \( \alpha^T R \alpha > 0 \). Furthermore, the optimal solution is given by \( \hat{x} = (\alpha^T R \alpha)^{-1} \alpha^T R \beta \).

From Lemma 4, the condition for minimization of (10) on \( x_{ik} \) can be characterized.

**Lemma 5.** For a fixed \( k \geq 0 \), consider the minimization problem in the quadratic game (5). There exists a unique minimal solution \( \{ \hat{z}_i \}_{i=0}^{\infty} \) if and only if \( \hat{x}^T \alpha_0 \hat{x} > 0 \), where \( \alpha_0 \) and \( \alpha \) are defined in (11). A necessary condition for unique minimum is that

\[
E_k^T W_{k-1}^{-1} E_k + H_k^T V_k^{-1} H_k - \gamma^2 L_k^T L_k > 0.
\]

A less stringent necessary condition for unique minimum is that \( E_k^T H_k^T \) has full row rank.

**Proof.** From (10) and Lemma 4, the minimization problem in the quadratic game (5) has a unique solution if and only if \( \hat{x}^T \alpha_0 \hat{x} > 0 \). It is easy to verify that \( \hat{x}^T \alpha_0 \hat{x} = E_k^T P_{k-1} E_k + H_k^T V_k^{-1} H_k - \gamma^2 L_k^T L_k \). Therefore, for \( k = 0 \) the condition (13) is necessary and unique minimum. For \( k > 0 \), we have that \( \hat{x}^T \alpha_0 \hat{x} \) can be written as

\[
\begin{bmatrix} E_k^T R_k \hat{R}_k \\ \alpha_0 & \hat{R}_k \hat{R}_k \end{bmatrix}.
\]

It follows that a necessary condition for \( \hat{x}^T \alpha_0 \hat{x} > 0 \) is that its (1, 1) block matrix is positive definite, that is,

\[
0 < E_k^T R_k \hat{R}_k = E_k^T W_{k-1}^{-1} E_k + H_k^T V_k^{-1} H_k - \gamma^2 L_k^T L_k.
\]

For a simpler necessary condition, note that from (13) we have

\[
E_k^T W_{k-1}^{-1} E_k + H_k^T V_k^{-1} H_k - \gamma^2 L_k^T L_k \geq 0.
\]

Remember that \( W_{-1} > 0 \) and \( V_k > 0 \) and therefore, it follows that \( E_k^T H_k^T \) must have full row rank.

**Remark 6.** The full row rank of \( E_k^T H_k^T \) is a necessary and sufficient condition for the existence of the standard descriptor Kalman filter (Ishihara, Terra, & Campos, 2005). The condition (13) is useful in order to determine a lower bound for \( \gamma \). The first player can always choose \( \hat{z}_i = L_i \hat{x}_i, i = 0, \ldots, k \). Therefore, at each instant \( k \), necessary and sufficient conditions for the existence of a solution \( \hat{x}_k \) for the \( H_\infty \) descriptor filter is that the minimization problem in the game (5)–(7) has a solution.

Now, from Lemma 5, it is clear that the successive \( H_\infty \) descriptor filtering problem has a solution if and only if we have

\[
\alpha_0^T \alpha_0 \alpha_0 > 0, \quad \alpha_{k-1}^T \alpha_{k-1} \alpha_0 > 0, \quad \ldots, \quad \alpha_0^T \alpha_0 \alpha_k > 0, \ldots
\]

Clearly, these existence conditions turns to have their examination impractical for crecent values of \( k \). Using the recurrent relations (12), it is possible to obtain a recursive test for the filter existence.

**Theorem 2.1.** The successive \( H_\infty \) filter problem is solvable if and only if \( P_{i+1,k} > 0, k = 0, 1, \ldots \), where the sequence \( \{ P_{i,k} \} \) is calculated by the following recursion

\[
P_{i,0}^{-1} := E_k^T P_{i,0}^{-1} E_k + H_k^T V_k^{-1} H_k - \gamma^2 L_k^T L_k
\]

\[
P_{i,k+1} := E_k^T (W_{i+1} + F_{i+1} P_{i+1,k+1} F_{i+1}^T)^{-1} E_k + H_k^T V_k^{-1} H_k - \gamma^2 L_k^T L_k.
\]

**Proof.** Define for \( k = 0 \), the auxiliary variable \( M_{0,i}^{-1} := \alpha_0^T \alpha_0 \alpha_0 \). For \( k > 0 \), using the recurrent relations (12), \( \alpha_i^T \alpha_i \alpha_k \) can be written as

\[
\begin{bmatrix} E_k^T R_k \hat{R}_k \\ \alpha_0 & \hat{R}_k \hat{R}_k \end{bmatrix}.
\]

In order to have \( \alpha_i^T \alpha_i \alpha_k > 0 \), the (2, 2) sub-block in (16) must be positive definite. Using the hypothesis that the filtering problem has been solved until the last step, the positiveness
of \( Q_{k-1}^{T} R_{k-1} Q_{k-1} \) is guaranteed and we only need to verify if 
\[ A_{k-1}^{T} R_{k-1} A_{k-1} = \begin{bmatrix} F_{k-1}^{T} Q_{k-1}^{-1} F_{k-1} & 0 \\ 0 & 0 \end{bmatrix} \geq 0. \]

Therefore, the (2, 2) term in (16) is positive definite and \( Q_{k}^{T} P_{k} Q_{k} > 0 \) if and only if the Schur complement of the (2, 2) block in (16) is positive definite. Define for \( k > 0 \), the auxiliary variable \( M_{k-1}^{T} \) as the Schur complement of the (2, 2) block in (16), or equivalently, \( M_{0:k} \) as the (1, 1) block of the inverse of (16).

\[ M_{k+1}^{T} := E_{k}^{T} (R_{k}^{-1} + \alpha_{k} (Q_{k}^{T} R_{k} Q_{k})^{-1} Q_{k}^{T} R_{k})^{-1} E_{k}. \]

Recalling that for \( k = 1 \), \( M_{0:k-1} \) is the (1, 1) block of \( (Q_{k}^{T} R_{k} Q_{k})^{-1} \). With \( \alpha_{k} \) defined in (12) we obtain
\[ M_{k+1}^{T} := E_{k}^{T} (R_{k}^{-1} + \alpha_{k} M_{k-1} A_{k-1}^{T} R_{k}^{-1})^{-1} E_{k}. \]  

(17)

Define \( P_{0:k} := M_{0:k} \). Writing \( P_{0:k} \) in terms of the original data (11), we obtain (14)-(15).

Theorem 2.2. Consider a sequence of measurements (1), the attenuation level \( \gamma \geq 0 \), and the sequence \( \{p_{k,i}\} \) calculated by the recursion (15). If the \( H_{\infty} \) estimation problem (5)-(7) is solvable for successive \( k \), a possible filtered estimate at each \( k \) is
\[ \hat{x}_{k|k} = L_{k} \hat{x}_{k} \]  

(18)

where \( \hat{x}_{k} \) is recursively computed as
\[ \hat{x}_{0|0} := (P_{0|0}^{-1} + \gamma^{-2} L_{0}^{T} S_{0}^{-1} L_{0})^{-1} (E_{0}^{T} P_{0|0}^{-1} F_{0} \hat{x}_{0} + H_{0}^{T} V_{0}^{-1} y_{0}), \]
\[ \hat{x}_{k|k} := (P_{k|k}^{-1} + \gamma^{-2} L_{k}^{T} S_{k}^{-1} L_{k})^{-1} \]
\[ \times (E_{k}^{T} X_{k-1} F_{k-1} - \gamma^{-2} L_{k}^{T} S_{k}^{-1} y_{k}) + H_{k}^{T} V_{k}^{-1} y_{k}. \]

(19)

Proof. Considering that the recursive \( H_{\infty} \) filter exists, for each \( k > 0 \) we obtain from Lemma 4 the solution of the minimization of (10) as
\[ \hat{x}_{k|k} = (\alpha_{k} Q_{k} R_{k} Q_{k})^{-1} \alpha_{k} Q_{k} R_{k} \hat{x}_{k}. \]

Based on the recurrent relations (11) and (12), considering \( \alpha_{0:k-1} \) and \( \hat{x}_{0:k-1} \) for \( j \geq k - 1 \), and introducing
\[ \begin{bmatrix} P_{0:k} \\ P_{2:k}^{T} \end{bmatrix} = \begin{bmatrix} E_{k}^{T} R_{k} E_{k} & \alpha_{k}^{T} Q_{k}^{-1} \alpha_{k-1} \\ \alpha_{k}^{T} R_{k} \alpha_{k-1} & \alpha_{k}^{T} \alpha_{k-1} + \alpha_{k}^{T} P_{k-1}^{-1} \alpha_{k-1} \end{bmatrix}^{-1} \]
we obtain the \( H_{\infty} \) estimate (19), taking into account that the first player chooses \( z_{k|k} = \hat{x}_{k|k} := L_{k} \hat{x}_{k} \).

Remark 7. With \( E_{k+1} = 1 \), Theorem 2.2 collapses to the a posteriori \( H_{\infty} \) recursions for state-space systems. It can be shown that the obtained recursions are equivalent to those of the standard \( H_{\infty} \) literature (for values of \( \gamma \) for which the a priori and a posteriori filters exist, it is only necessary to carry out some algebraic manipulations to obtain, for example, the expressions proposed in Hassibi et al. (1999), Shaked and Theodor (1992) and Theodor and Shaked (1994)). In the approach developed in this paper, the Riccati recursion is written only in terms of \( P_{k} \). Note that this kind of Riccati expression is useful since the \( H_{\infty} \) filter and the \( H_{\infty} \) predictor may not co-exist at the same \( \gamma \) level.

Remark 8. Notice that we obtain with the approach proposed in this paper, expressions to solve the one-step smoothing problem.

2.2. \( H_{\infty} \) descriptor predictor

In this subsection we present the \( H_{\infty} \) descriptor predictor. Since its derivation follows exactly the same line used for the derivation of the \( H_{\infty} \) descriptor filter of the previous subsection, the results will be presented without proof. The main difference is the relation between the auxiliary variable \( M_{k+1}^{T} \) and the Riccati variable \( P_{k+1}^{-1} \), as is presented in (24). As a counterpart of the Lemma 5, we can state the following lemma.

Lemma 9. For a fixed \( k \geq -1 \), consider the minimization problem

\[ E_{k+1}^{-1} P_{k+1}^{-1} E_{0} - \gamma^{-2} L_{k}^{T} S_{k}^{-1} L_{0} > 0, \quad k = -1 \]
\[ E_{k+1}^{-1} W_{k+1}^{-1} y_{k+1} - \gamma^{-2} L_{k+1}^{T} S_{k+1}^{-1} L_{k+1} > 0, \quad k \geq 0. \]

(20)

(21)

A less stringent necessary condition for a unique minimum is that \( E_{k} \) has full column rank.

Remark 10. The full column rank of \( E_{k} \) is a sufficient condition for the existence of the standard descriptor Kalman predictor (Ishihara et al., 2005). The conditions (20) and (21) are useful in order to determine a lower bound for \( \gamma \).

Theorem 2.3. The successive \( H_{\infty} \) descriptor prediction problem (8) is solvable if and only if \( P_{k+1}^{-1} \) is solvable for successive \( k \), and the sequence \( \{P_{k+1}^{-1}\} \) is calculated by the recursion
\[ P_{0|0} = (E_{0}^{T} P_{0|0}^{-1} E_{0})^{-1} \]
\[ P_{k+1} = (E_{k}^{T} \Omega_{k}^{-1} E_{k+1})^{-1} \Omega_{k} := W_{k} + F_{k} P_{k|k-1} F_{k}^{T} - F_{k} P_{k|k-1} \left[ H_{k} H_{k}^{T} \right] L_{k}^{-1} P_{k|k-1} F_{k} \]
\[ W_{\infty|k} := \begin{bmatrix} W_{k} & 0 \\ 0 & -\gamma^{2} S_{k} \end{bmatrix} + H_{k}^{T} \left[ H_{k} H_{k}^{T} \right] L_{k}^{-1} P_{k|k-1} F_{k} \]

(22)

and a possible solution is given by
\[ \hat{x}_{k+1|k} = L_{k} \hat{x}_{k} \]

(23)

where \( \hat{x}_{k} \) is recursively computed as
\[ \hat{x}_{k+1|k} = P_{k+1|k} E_{k}^{T} \Omega_{k}^{-1} F_{k} \hat{x}_{k|k} + P_{k+1|k} E_{k}^{T} \Omega_{k}^{-1} F_{k} \]
\[ \times M_{k+1|k} H_{k}^{T} \left[ (R_{k} + H_{k} M_{k+1|k} H_{k}^{T})^{-1} \right] L_{k} \]
\[ M_{k+1|k}^{-1} := P_{k+1|k}^{-1} - \gamma^{-2} L_{k} S_{k}^{-1} L_{k}. \]

(24)

Remark 11. With \( E_{k+1} = 1 \), Theorem 2.3 furnish exactly the standard \( H_{\infty} \) predictor recursions for state-space systems (cf. Hassibi et al. (1999), Shaked and Theodor (1992) and Theodor and Shaked (1994)).

Remark 12. For descriptor systems, the difference between filtering and prediction is more noticeable than in state space setting. In fact, if \( \gamma \) goes to infinity, the \( H_{\infty} \) estimators of Theorems 2.2 and 2.3 tend to the Kalman estimators. However, for descriptor systems, the existence of the a posteriori Kalman estimator does not assure the existence of the a priori Kalman estimator (Ishihara et al., 2005).

3. \( H_{\infty} \) descriptor information filters

The descriptor filter and predictor algorithms in information form compute the inverse of the covariance matrix, \( P_{k+1|k}^{-1} \) and \( P_{k+1|k}^{-1} \), termed information matrices, and compute the state information estimates \( P_{k}^{-1} \hat{x}_{k} \) and \( P_{k+1|k}^{-1} \hat{x}_{k+1|k} \) respectively. Applying the matrix
inversion lemma\(^1\) in Eqs. (15) and (19), the \(\mathcal{H}_\infty\) filtered information recursion is obtained as follows
\[
P_{k|k-1}^{\mathcal{H}_\infty} := P_{k|k-1} - \gamma^2 T_{k|k-1}^{-1} E_k T_{k|k-1} - 1 P_{k|k-1}^{-1} - 1 \left[ E_k T_{k|k-1} - 1 + E_k T_{k|k-1} - 1 P_{k|k-1}^{-1} - 1 \right] \left[ E_k T_{k|k-1} - 1 x_{k-1} - 1 \right]^{-1} + H_k^T V_k^{-1} y_k \tag{25}
\]
with the initial condition given by
\[
P_{0|0}^{\mathcal{H}_\infty} := P_{0|0} - \gamma^2 T_{0|0}^{-1} L_0^{-1} \times E_0 T_{0|0} - 1 x_0 + H_0^T V_0^{-1} y_0 \tag{26}
\]
where
\[
P_{0|0}^{-1} := E_0 T_{0|0} - 1 E_0 + H_0^T V_0^{-1} H_0 - \gamma^2 T_{0|0}^{-1} L_0^{-1} E_0 T_{0|0} - 1 \times E_0 T_{0|0} - 1 W_0^{-1} H_0 \tag{27}
\]
Similarly, from (22) and (24), applying the matrix inversion lemma and after some algebra, we obtain the recursive \(\mathcal{H}_\infty\) predicted information estimates (28) with the initial state given by
\[
P_{k|k+1}^{\mathcal{H}_\infty} := E_k T_{k|k-1} - 1 W_k^{-1} F_k \left( M_{k|k-1} - 1 E_k T_{k|k-1} - 1 + E_k T_{k|k-1} - 1 + H_k^T V_k^{-1} H_k \right)^{-1} \times E_k T_{k|k-1} - 1 W_k^{-1} F_k \left( P_{k|k-1}^{-1} - 1 - P_{k|k-1}^{-1} - 1 M_{k|k-1} - 1 \right) - 1 \left( E_k T_{k|k-1} - 1 x_{k-1} - 1 \right) \tag{28}
\]
where
\[
P_{0|0}^{-1} := E_0 T_{0|0} - 1 E_0 \tag{29}
\]

Remark 13. These \(\mathcal{H}_\infty\) information filters collapse to the filters developed in Terra et al. (2007b) when \(\gamma \to \infty\). If we set \(E_1 = I\), these expressions provide alternative estimates for the standard state-space information filters developed in Kailath, Sayed, and Hassibi (2000, Chapter 9, p. 322). For the information filters proposed in this paper, it is not necessary to invert the matrix \(F_i\).

Now, we are in a position to derive the square-root array algorithms for the filtered information and predicted information estimates.

4. \(\mathcal{H}_\infty\) array algorithms for information filters

Array algorithms have been used to avoid round-off errors that can occur in recursive Riccati equations which can cause the loss of positive-definiteness of the solutions. In essence, array algorithms reduce the dynamic range in fixed-point implementations and assure better condition numbers than the conventional Kalman filter algorithm. The algorithms we present in the following are based on some basic results that can be found in, Hassibi et al. (2000), Hassibi et al. (1999), Householder (1964) and Verhaegen and Van dooren (1986). In particular, it is important to detach the following theorem.

\[\text{Theorem 4.1 (Hassibi et al., 2000 (J-unitsaryarization))},\]
Let \(A\) and \(B\) be arbitrary \(n \times n\) and \(m \times m\) matrices, respectively, and suppose
\[
J = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}
\]
where \(S_1\) and \(S_2\) are \(n \times n\) and \(m \times m\) signature matrices.
Then \([A B]\) can be triangularized by a J-unitsary transform \(\Theta\) as
\[
[A B] \Theta = \begin{bmatrix} L & 0 \end{bmatrix},
\]
with \(L\) lower triangular, if and only if, all leading sub-matrices of \(S_1\) and of \(AS_1 A^T + BS_2 B^T\) have the same inertia.

Now we can present the square-root array algorithms for singular \(\mathcal{H}_\infty\) filtered and predicted recursions.

4.1. Filtered information

The filtered information algorithm computes the inverse \(P_{k|k}^{-1}\). In order to achieve an array algorithm that propagates the square-root factor \(P_{k|k}^{-1/2}\), we need to define a pre-array
\[
\begin{bmatrix}
P_{k-1|k-1}^{-1/2} F_k^{-1} W_{k-1}^{-1/2} & 0 \\ E_k^{-1} W_{k-1}^{-1/2} [L_k^T H_k^T] S_{v, k}^{-1/2} \\
\end{bmatrix}
\tag{29}
\]
where
\[
S_{v, k}^{-1/2} := \begin{bmatrix} y_k^{-1} S_k^{-1/2} & 0 \\ 0 & V_k^{-1/2} \end{bmatrix}
\tag{30}
\]
is the indefinite square root of \(S_{v, k}^{-1}\) with \(S = (-I) \oplus I\). By Theorem 4.1, considering \(J = I \oplus I \oplus (-I) \oplus I\), there exists a J-unitsary matrix \(\Theta_k\) that can triangularize the pre-array (29) if and only if \(J_1 = I \oplus I\) and the matrix \(C = AS_1 A^T + BS_2 B^T\) have the same inertia, with
\[
A := \begin{bmatrix} P_{k-1|k-1}^{-1/2} F_k^{-1} W_{k-1}^{-1/2} & 0 \\ 0 & E_k^{-1} W_{k-1}^{-1/2} \end{bmatrix}, \\
B := \begin{bmatrix} 0 \\ [L_k^T H_k^T] S_{v, k}^{-1/2} \end{bmatrix}
\]
and \(J_2 = (-I) \oplus I\). The necessary and sufficient conditions for the existence of \(\Theta_k\) at time \(k\) is that the right side of (27) should be positive definite and
\[
P_{k-1|k-1}^{-1} F_k^{-1} W_{k-1}^{-1} F_k^{-1} > 0.
\tag{31}
\]
If we suppose that the filter exists until the instant \(k - 1\), the condition (31) is redundant at the instant \(k\). Therefore the necessary and sufficient conditions for the existence of \(\Theta_k\) at the instant \(k\) is that the right side of (27) is positive definite. They are exactly the existence conditions of the \(\mathcal{H}_\infty\) filter, as shown in Theorem 2.1. This means that the \(\mathcal{H}_\infty\) descriptor filter exists if and only if the pre-array can be triangularized.

After some algebra it can be shown that the array algorithm for the filtered informations is given by
\[
\begin{bmatrix}
P_{k-1|k-1}^{-1/2} F_k^{-1} W_{k-1}^{-1/2} & 0 \\ E_k^{-1} W_{k-1}^{-1/2} [L_k^T H_k^T] S_{v, k}^{-1/2} \\
\end{bmatrix} \Theta_k \\
= \begin{bmatrix} P_{k-1|k-1}^{-1/2} F_k^{-1} W_{k-1}^{-1/2} & 0 \\ 0 & P_{k|k}^{-1/2} \end{bmatrix}
\tag{32}
\]
where \(P_{k|k} := P_{k-1|k-1} + F_k^{-1} W_{k-1}^{-1} F_k^{-1} > 0\).

4.2. Predicted information

Following the same procedure used to find the array algorithm for the filtered information estimate, a predicted information array

\[\text{\(A + BD C\)}^{-1} = A^{-1} - A^{-1} B (CA^{-1} B + D^{-1})^{-1} CA^{-1}.\]
algorithm can be expressed as

\[
\begin{bmatrix}
M_{k+1}^{-1/2} & 0 & 0 \\
0 & E_k^{-1/2} & 0 \\
0 & 0 & \gamma^{-1/2} I_k^{-1} S_{k+1}^{-1/2}
\end{bmatrix}
\begin{bmatrix}
\Theta_k \\
E_k F_k W_k \\
M_{k+1}^{-1/2}
\end{bmatrix}
= \begin{bmatrix}
M_{k+1}^{-1/2} & 0 & 0 \\
0 & E_k^{-1/2} & 0 \\
0 & 0 & M_{k+1}^{-1/2}
\end{bmatrix}
\begin{bmatrix}
\Theta_k \\
E_k F_k W_k M_{p,k}^{-1/2} \\
M_{k+1}^{-1/2}
\end{bmatrix}
\] (33)

where \(M_{k+1}^{-1} = P_{k+1}^{-1} - \gamma^{-1} L_{k+1} E_k S_k^{-1} L_k, M_{p,k} = M_{k+1}^{-1} + H_k W_k^{-1} H_k + F_k W_k F_k\).

**Remark 14.** For both singular array algorithms in filtered and predicted information forms (32) and (33), when \(E_{k+1} = I\) these algorithms collapse to the state-space forms. To the best of the authors' knowledge, for descriptor systems, there are no equivalent array algorithms in the literature to be compared with these results.

5. Conclusion

In this paper, the \(H_{\infty}\) recursive estimation problem for general time-variant descriptor systems is solved by applying data fitting and game theory approaches. Besides the usual necessary and sufficient \(H_{\infty}\) filter existence conditions based on the solution of a Riccati recursion, this paper has developed a new necessary condition for the existence of this kind of filter depending only on the original parameters of the system. This necessary test can be used to give a lower bound on the admissible value of the attenuation level \(\gamma\). For numerical proposals, we have also presented algorithmic procedures to implement a priori and a posteriori \(H_{\infty}\) filters in information form.

References


João Y. Ishihara is Associate Professor of electrical engineering at University of Brasilia, Brazil. He received his Ph.D. degree in Electrical Engineering in 1998 from the University of São Paulo. His current research interests include singular system theory, H2 control, robust control, and filtering.

Marco H. Terra is Associate Professor of electrical engineering at University of São Paulo (USP) at São Carlos, Brazil. He received his Ph.D. in electrical engineering in 1995 from USP. He is a member of IEEE. His research interests comprehend filtering and control theories, fault detection and isolation problems, and robotics.

B.M. Espinoza received the Engineer degree from the pontifical catholic university of Peru, Lima, Peru in 2001, and the M.Sc. degree from the University of Brasilia, Brazil, in 2004, both in electrical engineering. He is currently pursuing the Ph.D. degree at the Department of Electrical Engineering, University of Brasilia. His main research interests are in video coding, image processing, distributed source coding and optimal control.