Nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control applied to manipulators via actuation redundancy

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Abstract

This paper develops a new control algorithm based on nonlinear techniques, $\mathcal{H}_2$, $\mathcal{H}_\infty$, and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ via game theory for underactuated manipulators. These controllers are applied in an optimal way, via actuation redundancy, to minimize the energy consumption of the system. Results obtained from an actual manipulator robot are presented.

Keywords: Nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control; Manipulator robot; Redundancy and optimal control

1. Introduction

Nonlinear $\mathcal{H}_\infty$ control, based on game theory, has been applied to control robotic manipulators. This control methodology is useful for robust performance of manipulators under external disturbances (Chang & Chen, 1997; Chen, Lee, & Feng, 1994; Siqueira & Terra, 2002, 2004). More traditional design criterion is to achieve the $\mathcal{H}_2$ optimal control of robotic systems throughout the minimization of tracking errors and applied torques (Johansson, 1990). A nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design was proposed by Chen and Chang (1997) for tracking of rigid robotic systems under external disturbances. The design objective is to achieve the $\mathcal{H}_2$ optimal control under a desired $\mathcal{H}_\infty$ disturbance rejection constraint. To the authors’ knowledge, these controllers were never implemented in practice. The syntheses of these robust and optimal controllers were developed for manipulators with totally actuated joints. The main subject of this paper is to develop a position control algorithm for manipulators with unactuated (passive) joints based on actuation redundancy and nonlinear mixed $\mathcal{H}_2/\mathcal{H}_\infty$, $\mathcal{H}_\infty$, and $\mathcal{H}_2$ techniques. This algorithm develops a strategy in which active joints control passive joints, with a local redundancy resolution. It utilizes a coupling index as an optimization criterion to minimize the energy spent by the manipulator during the control of the unactuated joints. Maciel, Terra, and Bergerman (2003), solved this problem through feedback linearization and linear $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controls. The new approach proposed in this paper, based on nonlinear controllers (without feedback linearization), is motivated by the fact that it presents a unified design procedure where the control strategy is based on inertia and Coriolis matrices. Experimental results are included to demonstrate the effectiveness of this new approach. This paper is organized as follows: Section 2 presents the manipulator dynamic equation, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$, $\mathcal{H}_\infty$, and $\mathcal{H}_2$ controls for totally actuated robotic manipulators with experimental results and Section 3 presents the underactuated manipulator, the coupling index, the redundancy and optimal control for underactuated manipulators, the nonredundant control strategy, the state space dynamic equation of underactuated manipulators and the experimental results.

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2. Totally actuated manipulators

In this section, a suitable dynamic equation of the state tracking error for robotic manipulators and the controllers mixed $\mathcal{H}_2/\mathcal{H}_\infty$, $\mathcal{H}_\infty$, and $\mathcal{H}_2$ given by Chen and Chang (1997), Chen et al. (1994), Johansson (1990), respectively, are presented. The dynamic equation of a manipulator can be described by the Lagrange theory as

$$\tau = M(q)\ddot{q} + C(q, \dot{q})\dot{q} + F(\dot{q}) + G(q),$$

where $q \in \mathbb{R}^n$ are the joint positions, $M(q) \in \mathbb{R}^{n \times n}$ is the symmetric positive definite inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$ is the Coriolis and centripetal matrix, $F(\dot{q}) \in \mathbb{R}^n$ are the frictional torques, $G(q) \in \mathbb{R}^n$ are the gravitational torques, and $\tau \in \mathbb{R}^n$ are the applied torques. Parametric uncertainties can be introduced dividing the parameter matrices into a nominal and a perturbed part

$$M(q) = M_0(q) + \Delta M(q),$$
$$C(q, \dot{q}) = C_0(q, \dot{q}) + \Delta C(q, \dot{q}),$$
$$F(\dot{q}) = F_0(\dot{q}) + \Delta F(\dot{q}),$$
$$G(q) = G_0(q) + \Delta G(q).$$

Exogenous inputs, $\tau_d$, can also be introduced and Eq. (1) is rewritten as

$$\tau + \delta(q, \dot{q}, \ddot{q}, \tau_d) = M_0(q)\ddot{q} + C_0(q, \dot{q})\dot{q} + F_0(\dot{q}) + G_0(q)$$

with

$$\delta(q, \dot{q}, \ddot{q}, \tau_d) = - (\Delta M(q)\ddot{q} + \Delta C(q, \dot{q})\dot{q} + \Delta F(\dot{q}) + \Delta G(q) - \tau_d).$$

The dynamic equation of the state tracking error is defined as

$$\dot{x} = A_T(x, t)x + B_T(x, t)u + B_T(x, t)w$$

with

$$A_T(x, t) = T_0^{-1} \begin{bmatrix} -M_0^{-1}(q)C_0(q, \dot{q}) & 0 \\ T_{11}^{-1} & -T_{11}^{-1}T_{12} \end{bmatrix} T_0,$$
$$B_T(x, t) = T_0^{-1} \begin{bmatrix} M_0^{-1}(q) \\ 0 \end{bmatrix},$$
$$w = M_0(q)T_{11}M_0^{-1}(q)\delta(q, \dot{q}, \ddot{q}, \tau_d),$$

where the matrix $T_0$ is a state transformation defined as

$$T_0\dot{x} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \dot{q} - \dot{q}^d \\ \dot{q} - \dot{q}^d \end{bmatrix},$$

where $\dot{q}^d$ and $\dot{q}^d \in \mathbb{R}^n$ are the desired reference trajectory and the corresponding velocity, respectively; $T_{11}, T_{12} \in \mathbb{R}^{n \times n}$ are constant matrices to be determined. The control input selected, $u = M_0(q)T_1\dot{x} + C_0(q, \dot{q})T_1\dot{x}$ (where $T_1 = [T_{11}, T_{12}]$), is a selective applied torque, since it affects the kinetic energy only (Johansson, 1990).

The applied torque can be computed as follows:

$$\tau = M_0(q)\ddot{q} + C_0(q, \dot{q})\dot{q} + F_0(\dot{q}) + G_0(q)$$

with

$$\ddot{q} = \ddot{q}^d - T_{11}^{-1}T_{12}\dot{q} - T_{11}^{-1}M_0^{-1}(q)(C_0(q, \dot{q})B^TT_0\dot{x} - u).$$

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ nonlinear control aims to minimize a quadratic cost and to attenuate disturbances. The $\mathcal{H}_2$ optimal control should be obtained with the compromise of satisfying the $\mathcal{H}_\infty$ performance criterion. Given a desired disturbance level $\gamma > 0$ and weighting matrices $Q_1, Q_2,$ and $R$, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is solved if there exists a controller $u$ such that the $\mathcal{H}_2$ optimal performance

$$\min_{u(t)} J_2(u, w)$$

can be achieved under the $\mathcal{H}_\infty$ constraint

$$\max_{w(t) \in \mathbb{R}^n} J_2(u, w) < \bar{x}^T(0)P\bar{x}(0)$$

with

$$J_2(u, w) = \bar{x}^T(t)Q_2\bar{x}(t) + \int_0^{t_f} (\bar{x}^T(t)Q_2\bar{x}(t) + u^T(t)Ru(t)) \, dt,$$
$$J_1(u, w) = \bar{x}^T(t)Q_1\bar{x}(t) + \int_0^{t_f} (\bar{x}^T(t)Q_1\bar{x}(t) + u^T(t)Ru(t)) \, dt - \gamma^2 \int_0^{t_f} w^T(t)w(t) \, dt$$

where $P = P^T > 0$, $Q_{1f} = Q_{1f}^T > 0$, and $Q_{2f} = Q_{2f}^T > 0$. The solution for this problem is given in terms of the following coupled algebraic equations (Chen & Chang, 1997):

$$\begin{bmatrix} 0 & K_1 \\ K_1 & 0 \end{bmatrix} - T_0^TB\left( R - \frac{1}{\gamma^2}I_n \right) B^TT_0 + Q_1 = 0 \quad (6)$$
and

$$\begin{bmatrix} 0 & K_2 \\ K_2 & 0 \end{bmatrix} - T_0^TB\left( R - \frac{2}{\gamma^2}I_n \right) B^TT_0 + Q_2 = 0 \quad (7)$$

where $B = [I_n, 0]^T$. The optimal control and the worst disturbance can be written as

$$u^* = -R^{-1}B^TT_0\bar{x}; \quad w^* = \frac{1}{\gamma^2}B^TT_0\bar{x}.$$

To solve (6) and (7) some constraints are required for computing the matrices $Q_1, Q_2,$ and $R$. Subtracting (7) from (6) yields

$$\begin{bmatrix} 0 & K_1 - K_2 \\ K_1 - K_2 & 0 \end{bmatrix} - \frac{1}{\gamma^2}T_0^BB^TT_0 + Q_1 - Q_2 = 0.$$

Since $(1/\gamma^2)T^BB^TT$ is positive definite, the constraint, $Q_1 > Q_2 > 0$, must hold. The positive-definite
symmetric matrices $Q_1$ and $\Delta Q$ can be factorized as

$$Q_1 = \begin{bmatrix} Q_{11}^T Q_{11} & Q_{12}^T Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix};$$

$$\Delta Q = \begin{bmatrix} \Delta Q_{11}^T \Delta Q_{11} & \Delta Q_{12}^T \Delta Q_{12} \\ \Delta Q_{12} & \Delta Q_{22} \end{bmatrix}. $$

where $\Delta Q = Q_1 - Q_2$. The solutions of (6) and (7) are given by

$$T_0 = \begin{bmatrix} \gamma_1 \Delta Q_{11} & \gamma_1 \Delta Q_{22} \\ 0 & I_n \end{bmatrix}$$

(8)

$$K_1 = \frac{1}{2} (Q_{11}^T Q_{22} + Q_{22}^T Q_{11}) - \frac{1}{2} (Q_{12}^T Q_{12} + Q_{12}^T Q_{12})$$

(9)

and

$$K_2 = \frac{1}{4} (Q_{11}^T Q_{22} + Q_{22}^T Q_{11}) - \frac{1}{4} (Q_{12}^T Q_{12} + Q_{12}^T Q_{12})$$

(10)

where $K_1$ and $K_2$ are positive definite matrices $(Q_{11}^T Q_{22} + Q_{22}^T Q_{11} > Q_{12}^T Q_{12} + Q_{12}^T Q_{12})$. To guarantee the solvability of these two coupled equations, the matrix $R$ is of the form

$$R = \gamma_1^2 [I_n + (Q_{11} \Delta Q_{11}^T)(Q_{11} \Delta Q_{11}^T)]^{-1}.$$  

(11)

If only the $\mathcal{H}_\infty$ optimal design is considered, the cost functions (4) and (5) are combined into the following dynamic game problem (Chen et al., 1994):

$$\min_{u(t)} \max_{w(t) \in A_1 \times b_1, \xi(t)} J_1(u, w) \triangleq x^T(0)P x(0).$$

(12)

The simplified algebraic equation to solve this problem is given by (6), with the solution

$$T_0 = \begin{bmatrix} R_1^T Q_{11} & R_1^T Q_{22} \\ 0 & I_n \end{bmatrix}$$

(13)

and (9). The matrix $R_1$ is the result of the Cholesky factorization

$$R_1^T R_1 = \left( R^{-1} - \frac{1}{\gamma_1^2} I_n \right)^{-1}$$

where $R < \gamma^2 I_n$. For the $\mathcal{H}_2$ optimal design, the desired disturbance attenuation constraint (5) is not considered. The $\mathcal{H}_2$ optimal performance is defined by Johansson (1990):

$$\min_{u(t)} J_2(u, w).$$

Similar to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_\infty$ problems, the solution of the $\mathcal{H}_2$ problem is given in terms of a simplified algebraic equation

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} - T_0^T B R^{-1} B^T T_0 + Q_2 = 0$$

(14)

where $K = \frac{1}{2} (\hat{Q}_{11}^T \hat{Q}_{22} + \hat{Q}_{22}^T \hat{Q}_{11}) - \frac{1}{2} (\hat{Q}_{12}^T + \hat{Q}_{12})$,

$$Q_2 = \begin{bmatrix} Q_{11}^T & Q_{12}^T \\ Q_{12} & Q_{22} \end{bmatrix}, \quad T_0 = \begin{bmatrix} R_1^T \hat{Q}_{11} & R_1^T \hat{Q}_{22} \\ 0 & I_n \end{bmatrix}$$

and $R_1$ is defined by $R_1^T R_1 = R$.

2.1. Experimental results: totally actuated configuration

The nonlinear techniques described in this section were applied to the 3-link planar manipulator UArm II. It contains in each joint a DC motor, a brake and an optical encoder with quadrature decoding used to measure the joint positions. Joint velocities were obtained by numerical differentiation and filtering. Details of this planar manipulator, the matrices $M_0(q)$ and $C_0(q, \dot{q})$, and the kinematic and dynamic nominal parameters, was given by Siqueira and Terra (2004). The experiment was performed with initial positions $q(0) = [0^\circ, 0^\circ, 0^\circ]^T$ and desired final positions $q(T) = [-20^\circ, 30^\circ, -30^\circ]^T$, where the vector $t = [4.0, 4.0, 4.0]$ s contains the trajectory duration times for each joint. The reference trajectory, $q^d$, is a fifth-degree polynomial. Exogenous disturbances, starting at $t_d = 1.5$ s, were introduced in the form

$$\tau_d = \begin{bmatrix} -0.08 e^{-2(t-t_d)} \sin(2\pi(t-t_d)) \\ 0.04 e^{-2(t-t_d)} \sin(2\pi(t-t_d)) \\ -0.02 e^{-2(t-t_d)} \sin(2\pi(t-t_d)) \end{bmatrix}. $$

The following weighting matrices were defined for all controllers presented in Section 2:

$$R = \begin{bmatrix} 4.14 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3.86 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} J_3 & 0 \\ 0 & 4J_3 \end{bmatrix}.$$

The level of attenuation $\gamma$ was determined from (11) and $R < \gamma^2 I_n$, for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_\infty$ controllers, respectively. Table 1 shows the cost functions $J_1$ and $J_2$, the total energy (torque area) given by

$$E(t) = \sum_{i=0}^n \left( \int_0^t |\tau_i(t)| \, dt \right),$$

(15)
for all controllers, and the level of attenuation for the mixed $H_2/H_\infty$ and $H_\infty$ controllers. The experimental joint positions and applied torques are shown in Figs. 1–6. One can observe that the disturbance rejection constraint, $J_1 \leq 0$, is satisfied for both mixed $H_2/H_\infty$ and $H_\infty$ controllers. The greater $H_\infty$ controller capability to attenuate disturbances, represented by a lower value of $\gamma$, is confirmed by the lower value of the performance index $J_1$. However, the mixed $H_2/H_\infty$ controller presented the lowest energy consumption.

### 3. Underactuated manipulators

The dynamic equation of an underactuated manipulator can be obtained by partitioning (1) into components, corresponding to the active and passive
The control signal taking advantage of the actuation redundancy of the paper is controlling the positions of the passive joints, with passive joints dynamic coupling with the active joints, and are locked, the gradient and mixed

\[ \begin{bmatrix} \tau_a \\ 0 \end{bmatrix} = \begin{bmatrix} M_{aa}(q) & M_{aC}(q) \\ M_{Ca}(q) & M_{CC}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_c \end{bmatrix} + \begin{bmatrix} b_a(q, \dot{q}) \\ b_c(q, \dot{q}) \end{bmatrix}, \tag{16} \]

where \( b(q, \dot{q}) = C(q, \dot{q}) + F(\dot{q}) + G(q) \) and the subscripts \( a \) and \( u \) denote quantities related to the active and unlocked passive joints, respectively. It is considered that \( n_u \) joints of the manipulator described in (1) are unactuated, and the remaining \( n_a \) joints operate normally. When \( n_u > n_a \), one can define the following redundant control strategy: in a first control phase, the \( n_u \) passive joints are driven to the set-points via their dynamic coupling with the active joints, and are locked, and in a second control phase, the active joints are controlled.

3.1. Redundancy and optimal control for manipulators with passive joints

The main subject of the methodology proposed in this paper is controlling the positions of the passive joints, taking advantage of the actuation redundancy of the first control phase. For \( n_u > n_a \), the matrix \( M_{aa}(q) \) in (16) has dimension \( n_a \times n_u \). The second line of (16) can be rewritten as

\[ \ddot{q}_c = -\left( M^{-1}_{aa}(q) M_{aC}(q) \right) \dot{q}_a + b_a(q, \dot{q}) \]

\[ + (I - M^{-1}_{aa}(q) M_{aa}(q)) z, \tag{17} \]

where \((\cdot)^\dagger\) denotes the pseudo-inverse of a matrix, \( I \) is the identity matrix of order \( n_u \) and \( z \) is an arbitrary vector that can be selected as the gradient of a potential function, \( P(q) \)

\[ z = -k \left( \frac{\partial P(q)}{\partial q_a} \right)^T; \quad P(q) = -\sum_{j=1}^{n_u} \sigma_j(W_{au}(q)) \tag{18} \]

where \( k \) is a positive constant that represents the step of the gradient and

\[ W_{au}(q) = -(M_{au}(q) - M_{av}(q) M^{-1}_{av}(q) M_{av}(q))^{-1} \times M_{aa}(q) M^{-1}_{aa}(q) \]

is the relationship between the accelerations of the passive joints and the torques of the active joints. The open-loop relationship between \( \dot{q}_a \) and \( \tau_a \) is given by

\[ \tau_a = (M_{aa} - M_{av} M^{-1}_{av} M_{aa}) \dot{q}_a - M_{av} M^{-1}_{av} b_a \]

\[ + b_a + M_{aa}(I - M^{-1}_{aa} M_{aa})(-k) \left( \frac{\partial P(q)}{\partial q_a} \right)^T. \tag{19} \]

The control signal \( \dot{q}_a \) can be obtained by the nonlinear mixed \( H_2 / H_\infty, H_\infty, \) and \( H_2 \) controllers, described in Section 2, based on the model for underactuated manipulators presented in Section 3.3.

3.2. Nonredundant control strategy

The dynamic equation of the underactuated manipulator considering the nonredundant control strategy \( (n_u \) active joints control \( n_a \) passive joints and \( n_a - n_u \) active joints, in the first control phase) is given by Arai and Tachi (1991)

\[ \begin{bmatrix} \tau_a \\ 0 \end{bmatrix} = \begin{bmatrix} M_{aa}(q) & M_{aC}(q) \\ M_{Ca}(q) & M_{CC}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_c \end{bmatrix} + \begin{bmatrix} b_a(q, \dot{q}) \\ b_c(q, \dot{q}) \end{bmatrix} \tag{20} \]

where \( q_c \in \mathbb{R}^{n_c} \) contains \( n_a \) passive joints and \( n_c - n_u \) active joints and the vector \( q_c \in \mathbb{R}^{n_u} \) contains the remaining active joints. The relationship between the torque in the active joints and the acceleration of the joints \( q_c \) is given by

\[ \tau_a = (M_{aa}(q) - M_{av}(q) M^{-1}_{av}(q) M_{av}(q)) \ddot{q}_c \]

\[ + b_a(q, \dot{q}) - M_{av}(q) M^{-1}_{av}(q) b_a(q, \dot{q}). \tag{21} \]

3.3. State space dynamic equation for underactuated manipulators

To design the controllers presented in Section 2, the following dynamic equations for underactuated manipulators were developed based on (2)

\[ \begin{bmatrix} \tau_a \\ \tau_c \end{bmatrix} + \begin{bmatrix} \dot{q}_c \\ \dot{q}_c \end{bmatrix} = \begin{bmatrix} M_{cc}(q) & M_{cC}(q) \\ C_{cc}(q, \dot{q}) & C_{cc}(q, \dot{q}) \end{bmatrix} \begin{bmatrix} \ddot{q}_c \\ \ddot{q}_c \end{bmatrix} \]

\[ + \begin{bmatrix} F_c(q) \\ F_c(q) \end{bmatrix} + \begin{bmatrix} G_c(q) \\ G_c(q) \end{bmatrix} \tag{22} \]

where \( q_c \) is the vector of controlled joints and \( q_c \) is the vector of remaining joints. For the redundant case, \( q_c = q_u \) and \( q_c = q_u \); for the nonredundant case, \( q_c = q_u \) and \( q_c = q_t \). For the sake of simplicity, the index 0 representing the nominal system is dropped.

Consider the following state transformation:

\[ \mathbf{T}_0 \hat{\mathbf{x}}_c = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_c - \ddot{q}_c^d \\ q_c - q_c^d \end{bmatrix} \]

\[ \hat{\mathbf{x}}_c = \begin{bmatrix} \mathbf{T}_1 \hat{\mathbf{x}}_c + \mathbf{C}_c(q, \dot{q}) \mathbf{T}_1 \hat{\mathbf{x}}_c \end{bmatrix} \tag{24} \]

with \( \mathbf{T}_1 = [\mathbf{T}_{11}, \mathbf{T}_{12}] \), and the state space dynamic equation is defined by

\[ \hat{\mathbf{x}}_c = \mathbf{A}_T(\hat{\mathbf{x}}_c, t) \hat{\mathbf{x}}_c + \mathbf{B}_T(\hat{\mathbf{x}}_c, t) \mathbf{u} + \mathbf{B}_T(\hat{\mathbf{x}}_c, t) \mathbf{w} \tag{25} \]
respectively, where the vectors \(q_3\). The initial and final desired positions adopted were controlled in the first control phase. In the redundant nonredundant control strategy, joints 2 and 3 were selected properly by Maciel et al. (2003). In the (2004). For this configuration, two control phases are formed through APA configuration (joints 1 and 3 are active and joint 2 is passive) of the underactuated manipulator, UArm II, described by Siqueira and Terra. For the redundant case, \(\hat{q}_e = \hat{q}_d\) and the torque in the active joints are given by (26) and (19). For the nonredundant case, \(\hat{q}_i = \hat{q}_d\) and the torque in the active joints are given by (26) and (21).

3.4. Experimental results: underactuated configuration

The experiment presented in this section was performed through APA configuration (joints 1 and 3 are active and joint 2 is passive) of the underactuated manipulator, UArm II, described by Siqueira and Terra (2004). For this configuration, two control phases are necessary to control all joints to the set-point. Two control strategies were used: nonredundant and redundant with gradient step \(k = 0.01\) (this gradient step was selected properly by Maciel et al. (2003)). In the nonredundant control strategy, joints 2 and 3 were controlled in the first control phase. In the redundant strategy, only the passive joint 2 was controlled in the first control phase, applying torques in the active joints 1 and 3. The initial and final desired positions adopted were \(q(0) = [40^\circ - 30^\circ - 30^\circ]^T\), \(q(T^1, T^2) = [-40^\circ 30^\circ 10^\circ]^T\) respectively, where the vectors \(T^1\) and \(T^2\) contain the trajectory duration times for the control phases 1 and 2, respectively. For the nonredundant strategy, \(T^1 = [1.0 1.0]s\) and \(T^2 = [5.0]s\); for the redundant strategy, \(T^1 = [1.0]s\) and \(T^2 = [5.0 4.0]s\). External disturbances, starting at \(t_d = 0.3\) s, were introduced in the form

\[
\tau_d = \begin{bmatrix} 0.5e^{-4(t-t_d)} \sin(4\pi(t-t_d)) \\ -0.1e^{-6(t-t_d)} \sin(4\pi(t-t_d)) \\ 0.02e^{-6(t-t_d)} \sin(4\pi(t-t_d)) \end{bmatrix}
\]

for all experiments displayed in Figs. 7–20. The following weighting matrices were defined for all controllers of Section 3,

\[
Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 4.97 & 0 \\ 0 & 5.33 \end{bmatrix},
\]

for the nonredundant case,

\[
Q_2 = \begin{bmatrix} 0.55 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 0 \\ 0 & 13.75 & 0 \\ 0 & 0 & 0 & 0.50 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.55 & 0 \\ 0 & 13.75 \end{bmatrix}, \quad R = 4.97,
\]
for the redundant case, and

\[ Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 4.14 & 0 \\ 0 & 4 \end{bmatrix}, \]

\[ Q_2 = \begin{bmatrix} 0.15 & 0 & 0 & 0 \\ 0 & 0.20 & 0 & 0 \\ 0 & 0 & 2.4 & 0 \\ 0 & 0 & 0 & 0.20 \end{bmatrix} \]
for the second control phase, for both redundant and nonredundant cases.

Tables 2, 3, and 4 show the cost functions $J_1$, $J_2$, $E_\tau$, from (4), (5), and (15), for all controllers, and the level of attenuation $\gamma$ determined from (11) and $R<\gamma^2I_n$, for the mixed $H_2/H_\infty$ and $H_\infty$ controllers. The joint positions and applied torques are shown in Figs. 7–18.

One can observe from Table 4 the advantage of the redundant controls based on nonlinear controllers: the energies spent with the mixed $H_2/H_\infty$, $H_\infty$, and $H_2$ controllers were 15.23%, 16.03%, and 5.06%, respectively, smaller compared with the nonredundant control strategies. The disturbance rejection constraint, $J_1 \leq 0$, was satisfied for both mixed $H_2/H_\infty$ and $H_\infty$ controllers for the two control strategies, see Table 2. One can observe also, from Table 2, that it was payed a cost to economize energy, the index $J_1$ indicates the robustness of the mixed $H_2/H_\infty$ and $H_\infty$ controls, in the redundant case this index increases (that is consequence of less robustness to reject torque disturbances) for both controllers. The levels of attenuation $\gamma$ were the same for both control strategies, see Table 3.

For comparison purpose, the controller based on feedback linearization plus linear $H_\infty$ control presented...
by Maciel et al. (2003) was used to control the underactuated planar manipulator for the same initial and final positions, external disturbances (27), and redundant strategy (see Figs. 19 and 20). The acceleration formula given by the feedback linearization approach is defined as

$$q_c = q_c^d - K_p \ddot{q}_c - K_v \dot{q}_c - \bar{u},$$

where $K_p$ and $K_v$ are the proportional and derivative gains. One can observe that the acceleration formula in Eq. (26), given by nonlinear $H_\infty$ control via game theory

$$q_c = q_c^d - \mathbf{T}_{12}^{-1} \mathbf{T}_{12} \mathbf{q}_c - \mathbf{T}_{11}^{-1} \mathbf{M}_{cc}^{-1}(q)(C_{cc}(q, \dot{q})B^T \mathbf{T}_0 \mathbf{q}_c - \bar{u})$$

is different in nature. It involves, for example, the inertia and Coriolis matrices, $\mathbf{M}(q)$ and $C(q, \dot{q})$.

Computing the total energy spent by these controllers, torque areas of Figs. 16 and 20, the energy spent by the $H_\infty$ control based on feedback linearization is 0.180 N m s, approximately 15% greater than the total energy spent by the nonlinear $H_\infty$ control for the redundant case, see Table 4.

4. Conclusion

This paper presented an algorithm to control the position of unactuated joints of manipulators based on nonlinear $H_2$, $H_\infty$, and mixed $H_2/H_\infty$ control techniques. The control strategy shown uses a local redundancy resolution and a coupling index as an optimization criterion to minimize the energy spent by the manipulator. Experimental results show the advantage of this new approach.

References


