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Nonlinear $\mathcal{H}_\infty$ controllers for underactuated cooperative manipulators

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SUMMARY
In this paper, two nonlinear $\mathcal{H}_\infty$ control techniques are used to solve the position control problem of underactuated cooperative manipulators. The first technique consists in representing the nonlinear system in a quasi-linear parameter varying form and the solution is given in terms of linear matrix inequalities. The second technique gives an explicit solution to the cooperative manipulators $\mathcal{H}_\infty$ control problem. The control of the squeeze force between the manipulator end-effectors and the object is also evaluated. Results obtained from an actual cooperative manipulator, which is able to work as a fully actuated and an underactuated manipulator, are presented.

KEYWORDS: Nonlinear $\mathcal{H}_\infty$ control; Underactuated cooperative manipulators.

1. Introduction

Robotic systems composed of two or more manipulators transporting an object are denominated cooperative manipulators. This coupling among manipulators requires special attention, in position control problems, to the forces applied to the object. In ref. 1 a control paradigm for cooperative manipulators is proposed where the position and force controls are designed independently. The force is decomposed into motion force (generated by the system movement) and squeeze force. It is shown that only the squeeze force must be controlled, since the motion force goes to zero if the position control is stable.

On the other hand, cooperative manipulators as well as individual manipulators are subject to parametric uncertainties and external disturbances. Robust control approaches have been proposed in the literature for this kind of system to suppress undesired effects of these phenomena. In ref. 2 a semidecentralized adaptive fuzzy controller with $\mathcal{H}_\infty$ performance is developed for fully actuated cooperative manipulators, and simulated results illustrate the performance of this approach. The dynamic equation used in this reference is derived from the order reduction procedure proposed in ref. 3 for constraint manipulators.

This paper deals with the experimental validation of robust controllers for underactuated cooperative manipulators. One can define the underactuation in two possible ways: when it is caused by failures in the actuators, or when the manipulator is specifically designed taking into account the underactuation as a structural concept. The main question to be answered in this paper is related to the advantage of building underactuated cooperative manipulators. In this case, one can obtain structures that are lighter or less bulky. However, it is clear, mainly for this kind of robotic system, that the robustness against uncertainties and external disturbances naturally decreases if an appropriate robust control strategy is not applied. In order to avoid this problem, this paper shows the advantage of applying control strategies based on $\mathcal{H}_\infty$ approaches, taking into account an appropriate model for underactuated cooperative manipulators. It is important to observe that for this kind of robust control, the control inputs are minimized; consequently, the underactuated cooperative manipulator rejects disturbances with a minimum torque. Two nonlinear $\mathcal{H}_\infty$ control techniques based on centralized control strategies are evaluated in this paper: the $\mathcal{H}_\infty$ control for linear parameter varying (LPV) systems[4] and the $\mathcal{H}_\infty$ control based on game theory,[5] initially developed in the literature to control independent robotic arms. The details of these controllers, applied to the individual underactuated manipulator UArm II, can be seen in ref. 6. Here, the underactuated cooperative manipulator is composed of two manipulators connected to an object, Fig. 1. The experimental results obtained with the nonlinear $\mathcal{H}_\infty$ control techniques are compared based on three performance indexes that measure the tracking error of the object, the applied torque, and the squeeze force.

The robust control strategies proposed in this paper are also compared with the controller presented in refs. 7 and 8, where the hybrid position/force controller proposed in ref. 1 is extended to underactuated cooperative manipulators.

This paper is organized as follows: in Section 2, the dynamic equations for fully actuated and underactuated cooperative manipulators are presented, considering the squeeze force control proposed in refs. 1 and 7; in Section 3, the $\mathcal{H}_\infty$ control for LPV systems is presented; in Section 4, the quasi-LPV representation of underactuated cooperative manipulators is developed; in Section 5, the $\mathcal{H}_\infty$ control via game theory proposed in ref. 5 for manipulator robot

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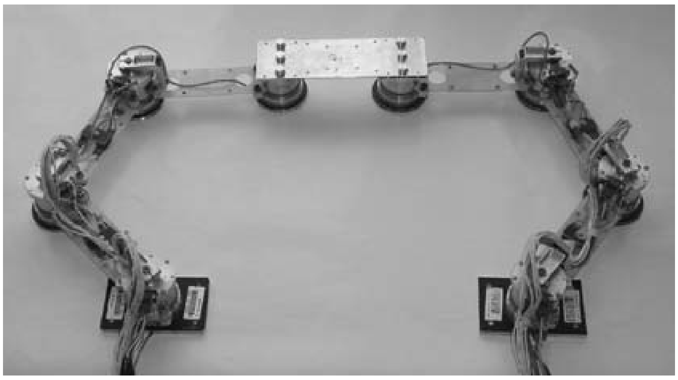

Fig. 1. Cooperative manipulator UArm II.

is presented; and in Section 6, the experimental results are displayed.

2. Cooperative Manipulators

2.1. Fully actuated cooperative manipulators

Consider a cooperative manipulator consisting of m fully actuated manipulators, each one with n degrees of freedom. Let $q_i \in \Re^n$ be the vector of generalized coordinates of manipulator i, and $x_o \in \Re^n$ the vector composed of Cartesian coordinates and orientation of the object, rigidly connected to the end-effectors of the manipulators. The geometric constraints generated by this configuration are given by $\varphi_i(x_o, q_i) = 0$ for $i = 1, 2, \ldots, m$. We denote $J_{o_i}(x_o, q_i)$ and $J_i(x_o, q_i)$ as the Jacobian matrices of $\varphi_i(x_o, q_i)$ with relation to x_o and q_i, respectively, i.e., $J_{o_i}(x_o, q_i) = \partial\varphi_i/\partial x_o$ and $J_i(x_o, q_i) = \partial\varphi_i/\partial q_i$. Hence, the velocity constraints are given by $\dot{\varphi}_i(x_o, q_i) = J_{o_i}(x_o, q_i)\dot{x}_o + J_i(x_o, q_i)\dot{q}_i = 0$ for $i = 1, 2, \ldots, m$. Assume that the relation

$$\dot{q}_i = -J_i^{-1}(x_o, q_i)J_{o_i}(x_o, q_i)\dot{x}_o$$

for $i = 1, 2, \ldots, m$ can always be computed. Then, the kinematic constraints are expressed by

$$\dot{\theta} = \begin{bmatrix} I_n \\ -J^{-1}(x_o)J_o(x_o) \end{bmatrix} \dot{x}_o \equiv B(x_o)\dot{x}_o \tag{1}$$

where $\theta = [x_o^T \quad q_1^T \ldots q_m^T]^T$, $J(x_o) = \mathrm{diag}[J_1(x_o, q_1), \ldots, J_m(x_o, q_m)]$, $J_o(x_o) = [J_{o_1}^T(x_o, q_1) \cdots J_{o_m}^T(x_o, q_m)]^T$, and $\mathrm{diag}[A, B, \ldots, Z]$ is a block-diagonal matrix composed of matrices $A, B, \ldots, Z$.

The dynamic equation of the object is given by

$$M_o(x_o)\ddot{x}_o + C_o(x_o, \dot{x}_o)\dot{x}_o + g_o(x_o) = J_o^T(x_o)h \tag{2}$$

where $M_o(x_o)$ is the inertia matrix, $C_o(x_o, \dot{x}_o)$ is the Coriolis and centripetal matrix, $g_o(x_o)$ is the gravitational torque vector, and $h = [h_i^T \cdots h_m^T]^T$ with $h_i \in \Re^n$ is the vector of applied force by the manipulator i in the object.

The dynamic equation of the manipulator i is given by

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i + J_i^T(x_o, q_i)h_i \tag{3}$$

where $M_i(q_i)$ is the inertia matrix, $C_i(q_i, \dot{q}_i)$ is the Coriolis and centripetal matrix, $g_i(q_i)$ is the gravitational torque vector, and τ_i is the applied torque vector, of manipulator i. Then, the dynamic equation of the cooperative manipulator can be represented as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = \begin{bmatrix} 0 \\ \tau \end{bmatrix} + \begin{bmatrix} J_o^T(x_o) \\ J^T(x_o) \end{bmatrix} h \tag{4}$$

where $M(\theta) = \mathrm{diag}[M_o(x_o), M_1(q_1), \ldots, M_m(q_m)]$, $C(\theta, \dot{\theta}) = \mathrm{diag}[C_o(x_o, \dot{x}_o), C_1(q_1, \dot{q}_1), \ldots, C_m(q_m, \dot{q}_m)]$, $g(\theta) = [g_o(x_o)^T \; g_1(q_1)^T \ldots g_m^T(q_m)]^T$, and $\tau = [\tau_1^T \ldots \tau_m^T]^T$.

Let h_o be the projection of h in the frame fixed on the center of mass of the object, $h_o = J_{oq}^T(x_o)h$, with $J_{oq}(x_o) = \mathrm{diag}[J_{o_1}(x_o, q_1), \ldots, J_{o_m}(x_o, q_m)]$. The resulting force in the object $h_{ro} = J_o^T(x_o)h$ can be rewritten as

$$h_{ro} = A^T J_{oq}^T(x_o)h = A^T h_o$$

where $A^T = [I_n \; I_n \cdots I_n] \in \Re^{n \times (nm)}$ and I_n is the identity matrix of size n. Since A^T is a nonsquare and a full row-rank matrix, there exists a nontrivial null space, denoted by squeeze subspace X_S, given by $X_S = \{h_{oS} \in \Re^{nm} | A^T h_{oS} = 0\}$. The dimension of X_S is $n(m-1)$. If h_o belongs to the null space X_S, the resulting force has no contribution to the object movement. It is defined the orthogonal decomposition of the projection of the applied force $h_o = h_{oS} + h_{oM}$, where h_{oS} is the projection of h_o in X_S, named squeeze force, and h_{oM} is the force induced by the system movement, named motion force. According to ref. 1, the matrix A^T is used to avoid unit inconsistency problems.

Considering the orthogonal decomposition, the dynamic equation of the cooperative manipulator, Eq. (4) can be represented as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = \tau_v + \overline{A}^T(x_o)h_{oS} \tag{5}$$

where τ_v is an auxiliary control input defined as

$$\tau_v = \begin{bmatrix} A^T h_{oM} \\ \tau + J^T(x_o)J_{oq}^{-T}(x_o)h_{oM} \end{bmatrix}$$

and $\overline{A}(x_o) = [A \;\; J_{oq}^{-1}(x_o)J(x_o)]$ is a Jacobian matrix.

If the auxiliary control input is partitioned in two vectors, $\tau_{v1} = A^T h_{oM}$ and $\tau_{v2} = \tau + J^T(x_o)J_{oq}^{-T}(x_o)h_{oM}$, the applied torque vector can be computed by

$$\tau = \tau_{v2} - J^T(x_o)J_{oq}^{-T}(x_o)(A^T)^+\tau_{v1} \tag{6}$$

where $(A^T)^+ = A(A^T A)^{-1}$ is the pseudoinverse of A^T. The motion force is given by $h_{oM} = (A^T)^+\tau_{v1}$. Hence, the control problem is to find an auxiliary control in order to guarantee stability and robustness against disturbances.

Considering the kinematic constraints in Eq. (1) and premultiplying the dynamic equation of the cooperative manipulator in Eq. (5) by $B^T(x_o)$ to eliminate the squeeze force term since $B^{\mathrm{T}}(x_o)\overline{A}^{\mathrm{T}}(x_o)=0$, one obtains

$$\overline{M}(x_o)\ddot{x}_o+\overline{C}(x_o,\dot{x}_o)\dot{x}_o+\overline{g}(x_o)=\overline{\tau}_v \tag{7}$$

where $\overline{M}(x_o)=B^{\mathrm{T}}(x_o)M(x_o)B(x_o)$, $\overline{C}(x_o,\dot{x}_o)=B^{\mathrm{T}}(x_o)M(x_o)\dot{B}(x_o)+B^{\mathrm{T}}(x_o)C(x_o,\dot{x}_o)B(x_o)$, $\overline{g}(x_o)=B^{\mathrm{T}}(x_o)g(x_o)$, and $\overline{\tau}_v=B^{\mathrm{T}}(x_o)\tau_v$.

2.2. *Underactuated cooperative manipulators*

Consider now that the joints of the cooperative manipulator are formed by n_a active joints (with actuators) and n_p passive joints (without actuators). The kinematic constraints in Eq. (1) can be rewritten as

$$\dot{\widetilde{\theta}}=\begin{bmatrix} I_n \\ -J_{AP}^{-1}(x_o)J_o(x_o)\end{bmatrix}\dot{x}_o\equiv\widetilde{B}(x_o)\dot{x}_o \tag{8}$$

where $\widetilde{\theta}=[x_o^{\mathrm{T}}\ q_a^{\mathrm{T}}\ q_p^{\mathrm{T}}]^{\mathrm{T}}$, $q_a\in\Re^{n_a}$ is the position vector of active joints, $q_p\in\Re^{n_p}$ is the position vector of passive joints, and $J_{AP}(x_o)$ is a Jacobian matrix generated from an orthogonal permutation matrix P_{AP}.[7] Therefore, if $\widetilde{q}=[q_a^{\mathrm{T}}\ \ q_p^{\mathrm{T}}]^{\mathrm{T}}=P_{AP}[q_1^{\mathrm{T}}\ \ q_2^{\mathrm{T}}\cdots q_m^{\mathrm{T}}]^{\mathrm{T}}$, then $J_{AP}(x_o)=[J_a(x_o)\ J_p(x_o)]=J(x_o)P_{AP}$.

The dynamic equation of underactuated cooperative manipulators can be given by

$$\widetilde{M}(\widetilde{\theta})\ddot{\widetilde{\theta}}+\widetilde{C}(\widetilde{\theta},\dot{\widetilde{\theta}})\dot{\widetilde{\theta}}+\widetilde{g}(\widetilde{\theta})=\begin{bmatrix}0\\ \tau_a\\ 0\end{bmatrix}+\begin{bmatrix}J_o^{\mathrm{T}}(x_o)\\ J_a^{\mathrm{T}}(x_o)\\ J_p^{\mathrm{T}}(x_o)\end{bmatrix}h \tag{9}$$

where $\widetilde{M}(\widetilde{\theta})=\mathrm{diag}[M_o(x_o),\ \ M_{AP}(\widetilde{q})]$, $M_{AP}(\widetilde{q})=P_{AP}\mathrm{diag}[M_1(q_1),\ldots,M_m(q_m)]P_{AP}^{\mathrm{T}}$, $\widetilde{C}(\widetilde{\theta},\dot{\widetilde{\theta}})=\mathrm{diag}[C_o(x_o,\dot{x}_o),\ C_{AP}(\widetilde{q},\dot{\widetilde{q}})]$, $C_{AP}(\widetilde{q},\dot{\widetilde{q}})=P_{AP}\mathrm{diag}[C_1(q_1,\dot{q}_1),\ldots,C_m(q_m,\dot{q}_m)]P_{AP}^{\mathrm{T}}$, $\widetilde{g}(\widetilde{\theta})=[g_o(x_o)^{\mathrm{T}}g_{AP}(\widetilde{q})^{\mathrm{T}}]^{\mathrm{T}}$, and $g_{AP}=P_{AP}[g_1^{\mathrm{T}}(q_1)\cdots g_2^{\mathrm{T}}(q_2)]^{\mathrm{T}}$. Note that the torques in the passive joints are zero, characterizing the underactuation.

Considering the orthogonal decomposition of the projection of the applied force $h=J_o^{-T}(x_o)A^{\mathrm{T}}(h_{oS}+h_{oM})$, Eq. (9) becomes

$$\widetilde{M}(\widetilde{\theta})\ddot{\widetilde{\theta}}+\widetilde{C}(\widetilde{\theta},\dot{\widetilde{\theta}})\dot{\widetilde{\theta}}+\widetilde{g}(\widetilde{\theta})=\tau_v+\widetilde{A}^{\mathrm{T}}(x_o)h_{oS} \tag{10}$$

where τ_v is an auxiliary control input defined as

$$\tau_v=\begin{bmatrix}A^{\mathrm{T}}h_{oM}\\ \tau_a+J_a^{\mathrm{T}}(x_o)J_{oq}^{-T}(x_o)h_{oM}\\ J_p^{\mathrm{T}}(x_o)J_{oq}^{-T}(x_o)h_{oM}\end{bmatrix}$$

and $\widetilde{A}(x_o)=[A\ \ J_{oq}^{-1}(x_o)J_a(x_o)\ \ J_{oq}^{-1}(x_o)J_p(x_o)]$ is a Jacobian matrix. If the auxiliary control input is partitioned in three vectors, $\tau_{v1}=A^{\mathrm{T}}(x_o)h_{oM}$, $\tau_{v2}=\tau_a+J_a^{\mathrm{T}}(x_o)\ \ J_{oq}^{-T}(x_o)h_{oM}$, and $\tau_{v3}=J_p^{\mathrm{T}}(x_o)J_{oq}^{-T}(x_o)h_{oM}$, the applied torque in the active joints can be computed as

$$\tau_a=\tau_{v2}-J_a^{\mathrm{T}}(x_o)J_{oq}^{-T}(x_o)\begin{bmatrix}A^{\mathrm{T}}\\ J_p^{\mathrm{T}}(x_o)J_{oq}^{-T}(x_o)\end{bmatrix}^{+}\begin{bmatrix}\tau_{v1}\\ \tau_{v3}\end{bmatrix}. \tag{11}$$

Considering the kinematic constraints in Eq. (8) and premultiplying Eq. (10) by $\widetilde{B}^{\mathrm{T}}(x_o)$ to eliminate the squeeze force term since $\widetilde{B}^{\mathrm{T}}(x_o)\widetilde{A}^{\mathrm{T}}(x_o)=0$, the dynamic equation of the underactuated cooperative manipulator is given by

$$\widetilde{M}(x_o)\ddot{x}_o+\widetilde{C}(x_o,\dot{x}_o)\dot{x}_o+\widetilde{g}(x_o)=\widetilde{\tau}_v \tag{12}$$

where $\widetilde{M}(x_o)=\widetilde{B}^{\mathrm{T}}(x_o)\widetilde{M}(\widetilde{\theta})\widetilde{B}(x_o)$, $\widetilde{C}(x_o,\dot{x}_o)=\widetilde{B}^{\mathrm{T}}(x_o)(\widetilde{M}(\widetilde{\theta})\dot{\widetilde{B}}(x_o)+\widetilde{C}(\widetilde{\theta},\dot{\widetilde{\theta}})\widetilde{B}(x_o))$, $\widetilde{g}(x_o)=\widetilde{B}^{\mathrm{T}}(x_o)\widetilde{g}(\widetilde{\theta})$, and $\widetilde{\tau}_v=\widetilde{B}^{\mathrm{T}}(x_o)\tau_v$.

From the control paradigm introduced in ref. 1 for cooperative manipulators, the position and squeeze force control problems can be decomposed and solved independently. In this case, the applied torque can be computed by

$$\tau=\tau_P+\tau_S$$

where τ_P are torques generated by the position control and τ_S are torques generated by the squeeze force control. In this paper, τ_P are given by Eq. (6) for fully actuated manipulators. For underactuated manipulators, $\tau_P=P_{AP}^{-1}[\tau_a^{\mathrm{T}}\ 0]^{\mathrm{T}}$, with τ_a given by Eq. (11). In Sections 4 and 5, the dynamic equations (7) and (12) are used to design robust controllers for position control of cooperative manipulators, considering parametric uncertainties and external disturbances in the manipulator and the object.

2.3. *Squeeze force control*

For the squeeze force control, ref. 1 proposed the utilization of an integral controller. For fully actuated manipulators, the applied torque related to the squeeze force control is given by

$$\tau_S=D^{\mathrm{T}}(x_o)\left[h_{oS}^{d}+K_i\int\left(h_{oS}^{d}-h_{oS}\right)\mathrm{d}t\right] \tag{13}$$

where h_{oS}^{d} is the desired squeeze force, K_i is a positive definite matrix, and

$$D(x_o)=\begin{bmatrix}J_{o_1}^{-1}(x_o,\Omega_1(x_o))J_1(x_o,\Omega_1(x_o)) & \ldots & 0\\ \vdots & \ddots & \vdots\\ 0 & \ldots & J_{o_k}^{-1}(x_o,\Omega_k(x_o))J_k(x_o,\Omega_k(x_o))\end{bmatrix}.$$

The dimension of h_{oS} is nm and, since the dimension of X_S is $n(m-1)$, it is possible to write $h_{oS}=\widehat{A}^{\mathrm{T}}\lambda_S$, where $\lambda_S\in\Re^{n(m-1)}$ and $\widehat{A}^{\mathrm{T}}\in\Re^{(nm)\times(n(m-1))}$ is the full rank matrix that projects the null space of A^{T} ($Im(\widehat{A}^{\mathrm{T}})=X_S$). Hence, vector λ_S represents the variables to be controlled. For the underactuated cooperative manipulator, (13) can be partitioned as

$$\begin{bmatrix}\tau_{Sa}\\ 0\end{bmatrix}=\begin{bmatrix}D_a^{\mathrm{T}}(x_o)\\ D_p^{\mathrm{T}}(x_o)\end{bmatrix}\widehat{A}^{\mathrm{T}}\lambda_S \tag{14}$$

where $[D_a(x_o)\ D_p(x_o)]=D(x_o)P_{AP}$. Note that n_p constraints are imposed in the components of λ_S since it is not possible to apply torque in the passive joints, $\tau_{Sp}=0$. As the manipulator robots considered here are nonredundant ones, not all components of λ_S can be independently controlled, (see ref. 7 for more details).

The vector λ_S is partitioned in the independently controlled components $\lambda_{Sc} \in \Re^{n_e}$, where $n_e = n(m-1) - n_p$ if $n_a > n$, and $n_e = 0$ if $n_a < n$, and in the not controlled components $\lambda_{Sn} \in \Re^{n_p}$. Note that if $n_a < n$, none of the components of λ_S can be controlled. The squeeze force controller is given by

$$\lambda_{Sc} = \left[\lambda^d_{Sc} + K_{i_S} \int \left(\lambda^d_{Sc} - \lambda_{Sc}\right) \mathrm{d}t\right] \tag{15}$$

where $\lambda^{\mathrm{d}}_{Sc}$ is the desired value for λ_{Sc} and K_{i_S} is a positive definite matrix. λ_{Sn} is computed from the constraints in Eq. (14) as a function of λ_{Sc}. The applied torque in the active joints related to the squeeze force control is given by Eq. (14) as

$$\tau_{Sa} = D^{\mathrm{T}}_{\mathrm{a}}(x_o)\widehat{A}^{\mathrm{T}}\lambda_{Sc}. \tag{16}$$

3. $\mathcal{H}_\infty$ Control of LPV Systems

Consider the following LPV system

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & 0 & 0 \\ C_2(\rho) & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{17}$$

where x is the state vector, u is the control input vector, w is the external input vector, z_1 and z_2 are the output variables, and ρ is the parameter varying vector. The system in Eq. (17) has $\mathcal{L}_2$ gain $\leq \gamma$ if

$$\int_0^{\mathrm{T}} \|z\|^2 \,\mathrm{d}t \leq \gamma^2 \int_0^{\mathrm{T}} \|w\|^2 \,\mathrm{d}t \tag{18}$$

for all $T \geq 0$ and all $w \in \mathcal{L}_2(0, T)$, with $x(0) = 0$. Assume that the underlying parameter ρ varies in the allowable set

$$F^\nu_P = \left\{\rho \in \mathcal{C}^1(\Re^+, \Re^k) : \rho \in P,\ |\dot{\rho}_i| \leq \nu_i\right\}$$

for $i = 1, \ldots, k$, where $P \subset \Re^k$ is a compact set, and $\nu = [\nu_1 \cdots \nu_k]^{\mathrm{T}}$ with $\nu_i \geq 0$. The $\mathcal{H}_\infty$ controller for LPV systems adopted in this paper is based on state feedback approach developed in ref. 4: If there exists a continuously differentiable matrix function $X(\rho) > 0$ that satisfies

$$\begin{bmatrix} E(\rho) & X(\rho)C^{\mathrm{T}}_1(\rho) & B_1(\rho) \\ C_1(\rho)X(\rho) & -I & 0 \\ B^{\mathrm{T}}_1(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{19}$$

where

$$E(\rho) = -\sum_{i=1}^{m} \pm \left(\nu_i \frac{\partial X}{\partial \rho_i}\right) - B_2(\rho)B^{\mathrm{T}}_2(\rho) + \widehat{A}(\rho)X(\rho) + X(\rho)\widehat{A}(\rho)^{\mathrm{T}}$$

and $\widehat{A}(\rho) = A(\rho) - B_2(\rho)C_2(\rho)$, then, the closed loop system has $\mathcal{L}_2$ gain $\leq \gamma$ under the state feedback law

$$u = -(B_2(\rho)X^{-1}(\rho) + C_2(\rho))x.$$

Hence, it is required to solve a set of parametric linear matrix inequalities (LMIs) represented by Eq. (19), which is an infinite convex optimization problem. A practical scheme based on basis functions for $X(\rho)$ and on gridding the parameter set P, also developed in ref. 4, is adopted here to solve these LMIs.

4. Quasi-LPV Representation of Cooperative Manipulators

Nonlinear systems can always be represented as LPV systems.[9] However, in this case, the parameter ρ in Eq. (17) is not only a function of time, but also of system states. This fact imposes a restrictive characteristic on the controller, since the compact set P and the parameter variation rate bounds ν, used in the design procedure, are not exactly known before the controller implementation. The values for P and ν must be checked during this phase. Due to this characteristic, this kind of representation is denominated quasi-linear parameter varying (quasi-LPV).

In this section, quasi-LPV representations of fully actuated and underactuated cooperative manipulators is developed, based on the following dynamic equation

$$\widehat{M}_0(x_o)\ddot{x}_o + \widehat{C}_0(x_o, \dot{x}_o)\dot{x}_o + \widehat{g}_0(x_o) + \widehat{\tau}_d = \widehat{\tau}_v \tag{20}$$

where $\widehat{M}_0(x_o) = \overline{M}_0(x_o)$, $\widehat{C}_0(x_o, \dot{x}_o) = \overline{C}_0(x_o, \dot{x}_o)$, $\widehat{g}_0(x_o) = \overline{g}_0(x_o)$, and $\widehat{\tau}_v = \overline{\tau}_v$ if all manipulators are fully actuated [Eq. (7)]; or $\widehat{M}_0(x_o) = \widetilde{M}_0(x_o)$, $\widehat{C}_0(x_o, \dot{x}_o) = \widetilde{C}_0(x_o, \dot{x}_o)$, $\widehat{g}_0(x_o) = \widetilde{g}_0(x_o)$, and $\widehat{\tau}_v = \widetilde{\tau}_v$ if any of the manipulators is underactuated, [Eq. (12)]. The index 0 indicates nominal values for the matrices and vectors. $\widehat{\tau}_d$ represents the vector of parametric uncertainties and external disturbances of the system.

The state tracking error is defined as

$$\widetilde{x} = \begin{bmatrix} \dot{x}_o - \dot{x}^d_o \\ x_o - x^d_o \end{bmatrix} = \begin{bmatrix} \dot{\widetilde{x}}_o \\ \widetilde{x}_o \end{bmatrix} \tag{21}$$

where x^d_o and $\dot{x}^d_o \in \Re^n$ are the desired reference trajectory and velocity of the object, respectively. The quasi-LPV representation of cooperative manipulators is found using Eqs. (20) and (21) as

$$\dot{\widetilde{x}} = A(x_o, \dot{x}_o)\widetilde{x} + Bu + Bw \tag{22}$$

with $w = \widehat{M}^{-1}_0(x_o)\widehat{\tau}_d$, $B = [I^{\mathrm{T}}_n \quad 0^{\mathrm{T}}]^{\mathrm{T}}$, and

$$A(x_o, \dot{x}_o) = \begin{bmatrix} -\widehat{M}^{-1}_0(x_o)\widehat{C}_0(x_o, \dot{x}_o) & 0 \\ I_n & 0 \end{bmatrix}.$$

From this equation, the variable $\widehat{\tau}_v$ can be represented as

$$\widehat{\tau}_v = \widehat{M}_0(x_o)\left(\ddot{x}^d_o + u\right) + \widehat{C}_0(x_o, \dot{x}_o)\dot{x}^d_o + \widehat{g}_0(x_o).$$

Although the matrix $\widehat{M}_0(x_o)$ explicitly depends on the object position x_o, one can consider it as a function of the position error $\widetilde{x}_o$. The same can be observed for $\widehat{C}_0(x_o, \dot{x}_o)$. Hence, Eq. (22) is a quasi-LPV representation of fully actuated and underactuated cooperative manipulators.

5. Nonlinear $\mathcal{H}_\infty$ Control via Game Theory

In this section, the game theory is used to solve the $\mathcal{H}_\infty$ control problem of cooperative manipulators. This solution is based on the results presented in ref. 5. From Eq. (21), after the state transformation given by

$$\widetilde{z} = \begin{bmatrix} \widetilde{z}_1 \\ \widetilde{z}_2 \end{bmatrix} = T_0 \widetilde{x} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\widetilde{x}}_o \\ \widetilde{x}_o \end{bmatrix} \tag{23}$$

where $T_1 = [T_{11} \quad T_{12}]$ and $T_{11}, T_{12} \in \Re^{n \times n}$ are constant matrices to be determined, the dynamic equation of the state tracking error becomes

$$\dot{\widetilde{x}} = A_T(\widetilde{x}, t)\widetilde{x} + B_T(\widetilde{x}, t)u + B_T(\widetilde{x}, t)w \tag{24}$$

with $w = \widehat{M}_0(x_o) T_{11} \widehat{M}_0^{-1}(x_o) \widehat{\tau}_d$

$$A_T(\widetilde{x}, t) = T_0^{-1} \begin{bmatrix} -\widehat{M}_0^{-1}(x_o)\widehat{C}_0(x_o, \dot{x}_o) & 0 \\ T_{11}^{-1} & -T_{11}^{-1}T_{12} \end{bmatrix} T_0$$

$$B_T(\widetilde{x}, t) = T_0^{-1} \begin{bmatrix} \widehat{M}_0^{-1}(x_o) \\ 0 \end{bmatrix}.$$

The relationship between the auxiliary control input $\widehat{\tau}_v$ and the control input u is given by

$$\widehat{\tau}_v = \widehat{M}_0(x_o)\ddot{x}_o^c + \widehat{C}_0(x_o, \dot{x}_o)\dot{x}_o + \widehat{g}_0(x_o) \tag{25}$$

with $\ddot{x}_o^c = \ddot{x}_o^d - T_{11}^{-1} T_{12} \dot{\widetilde{x}}_o - T_{11}^{-1} \widehat{M}_0^{-1}(x_o)(\widehat{C}_0(x_o, \dot{x}_o) B^T T_0 \widetilde{x} - u)$.

The $\mathcal{H}_\infty$ controller presented in this section solves the following minimax control problem[5]

$$\min_{u \in \mathcal{L}_2} \max_{0 \neq w \in \mathcal{L}_2} \frac{\int_0^\infty \left(\frac{1}{2}\widetilde{x}^T Q \widetilde{x} + \frac{1}{2} u^T R u\right) dt}{\int_0^\infty \left(\frac{1}{2} w^T w\right) dt} \le \gamma^2 \tag{26}$$

where Q and R are positive definite symmetric weighting matrices, γ is a desired disturbance attenuation level, $\widetilde{x}(0) = 0$, and $u = F(\tilde{x})\tilde{x}$. According to the game theory, the solution of this minimax problem is found if there exist matrices T_0 and $K > 0$ satisfying the following algebraic matrix equation

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} - T_0^T B \left(R^{-1} - \frac{1}{\gamma^2} I \right) B^T T_0 + Q = 0. \tag{27}$$

The solution of this equation is given in ref. 5. The control input is computed as

$$u = -R^{-1} B^T T_0 \tilde{x}. \tag{28}$$

Table I. Object parameters.

Mass, m_o (kg)	0.025
Length, l_o (m)	0.092
Center of mass, a_o (m)	0.046
Inertia, I_o(kg m)^3	0.000023

6. Experimental Results

To validate the proposed nonlinear $\mathcal{H}_\infty$ control solutions, they are applied to the underactuated cooperative manipulator of Fig. 1, composed of two planar underactuated manipulators Underactuated Arm II (UArm II). The kinematic and dynamic parameters of this manipulator can be found in ref. 6. The object parameters are presented in Table I.

A straight line in the plane XY, with a given orientation, is defined as a desired trajectory to move the center of mass of the object from $x_o(0) = [0.20\,\text{m} \;\; 0.35\,\text{m} \;\; 0°]^T$ to $x_o^d(T) = [0.25\,\text{m} \;\; 0.40\,\text{m} \;\; 0°]^T$, where $T = 5.0\,\text{s}$ is the trajectory duration time. The reference trajectory, $x_o^d(t)$, is a fifth-degree polynomial. The following external disturbances were introduced to verify the robustness of the proposed controllers

$$\tau_{d_1} = \begin{bmatrix} 0.01 e^{-\frac{(t-2.5)^2}{8}} \sin(4\pi t) \\ -0.01 e^{-\frac{(t-2.5)^2}{8}} \sin(5\pi t) \\ -0.01 e^{-\frac{(t-2.5)^2}{8}} \sin(6\pi t) \end{bmatrix} \quad \text{and}$$

$$\tau_{d_2} = \begin{bmatrix} 0.02 e^{-\frac{(t-2.5)^2}{8}} \sin(4\pi t) \\ 0.02 e^{-\frac{(t-2.5)^2}{8}} \sin(5\pi t) \\ 0.01 e^{-\frac{(t-2.5)^2}{8}} \sin(6\pi t) \end{bmatrix}.$$

The gains of the integral controllers for the squeeze force control are $K_i = 0.9 I_3$ and $K_{i_s} = 0.9 I_3$ for the fully actuated and underactuated cases, respectively.

6.1. *Fully actuated configuration*

To apply the algorithm described in Section 3, the manipulator needs to be represented as in Eq. (17). The parameter $\rho(\tilde{x})$ chosen is composed of the position and orientation errors, that is, $k = 3$ and $\rho(\widetilde{x}) = \widetilde{x}_o$. The following quasi-LPV system matrices are considered

$$\begin{aligned} A(\rho(x)) &= A(\rho(\widetilde{x})) \\ B_1(\rho(x)) &= B \\ B_2(\rho(x)) &= B \\ C_1(\rho(x)) &= I_6 \\ C_2(\rho(x)) &= 0 \end{aligned}$$

where $A(\rho(\widetilde{x}))$ and B are obtained from Eq. (22) with $\widehat{M}(x_o) = \overline{M}(x_o)$ and $\widehat{C}(x_o, \dot{x}_o) = \overline{C}(x_o, \dot{x}_o)$.

The compact set P is defined as $\rho \in [-0.1, 0.1]\,\text{m} \times [-0.1, 0.1]\,\text{m} \times [-9, 9]°$. The parameter variation rate is bounded by $|\dot{\rho}| \le [0.06\,\text{m/s} \;\; 0.06\,\text{m/s} \;\; 6°/\text{s}]$. The basis functions for $X(\rho)$ are selected as: $f_1(\rho(\widetilde{x})) = 1$, $f_2(\rho(\widetilde{x})) = \widetilde{x}_{o_X}$, $f_3(\rho(\widetilde{x})) = \widetilde{x}_{o_Y}$, and $f_4(\rho(\widetilde{x})) = cos(\widetilde{x}_{o_\phi})$, where $\widetilde{x}_o = [\widetilde{x}_{o_X} \; \widetilde{x}_{o_Y} \; \widetilde{x}_{o_\phi}]$, $\widetilde{x}_{o_X}$ and $\widetilde{x}_{o_Y}$ are the X and Y coordinate errors of the object, respectively, and $\widetilde{x}_{o_\phi}$ is the orientation error.

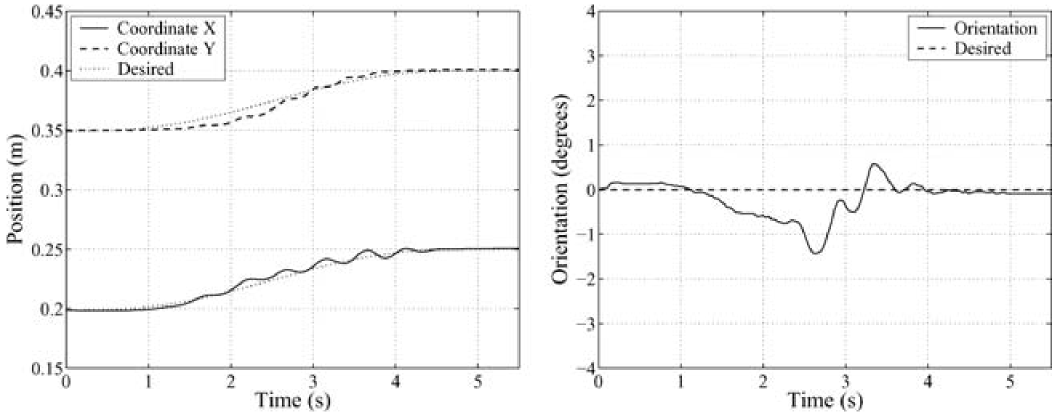

Fig. 2. Fully actuated configuration, control via quasi-LPV representation.

The parameter space was divided considering three points in the set P. The best attenuation level found was $\gamma = 1.25$.

For the nonlinear $\mathcal{H}_\infty$ control designed via game theory, described in Section 5, the attenuation level found was $\gamma = 4.0$. The weighting matrices used were $Q_1 = I_3$, $Q_2 = 10I_3$, $Q_{12} = 0$, and $R = I_3$. The desired values for the squeeze force are $h_{oS}^{d} = [0\ 0\ 0]^{\mathrm{T}}$. The experimental results, Cartesian coordinates, and orientation of the object, are shown in Figs. 2 and 3.

Three performance indexes are used to compare the nonlinear $\mathcal{H}_\infty$ controllers: the norm of the state vector

$$\mathcal{L}_2[\widetilde{x}] = \left(\frac{1}{(t_r)} \int_0^{t_r} \|\widetilde{x}(t)\|_2^2 \, \mathrm{d}t \right)^{\frac{1}{2}}$$

where $\|\cdot\|_2$ is the Euclidean norm; the sum of the applied torque by the ith joint for both manipulators

$$E[\tau] = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \left(\int_0^{t_r} |\tau_{j_i}(t)| \, \mathrm{d}t \right) \right)$$

and the sum of the squeeze force

$$E[h_{oS}] = \sum_{i=1}^{nm} \left(\int_0^{t_r} |h_{oS_i}(t)| \, \mathrm{d}t \right)$$

where t_r is the spent time for the object to reach the desired position. The results presented in Tables II and III are the average of the respective experiments run five times.

Table II. Performance indexes—fully actuated configuration.

Nonlinear $\mathcal{H}_\infty$	$\mathcal{L}_2[\widetilde{x}]$	$E[\tau]$ (N m s)	$E[h_{oS}]$ (N s)
Quasi-LPV	0.01815	0.8318	0.2193
Game theory	0.01158	1.1200	0.3875

Table III. Performance indexes—underactuated configuration.

Nonlinear $\mathcal{H}_\infty$	$\mathcal{L}_2[\widetilde{x}]$	$E[\tau]$ (N m s)	$E[h_{oS}]$ (N s)
Quasi-LPV	0.0154	0.9976	0.4477
Game theory	0.0103	1.0609	0.3973

Table II shows the values of $\mathcal{L}_2[\widetilde{x}]$, $E[\tau]$, and $E[h_{oS}]$ computed with the results obtained from the implementation of the nonlinear $\mathcal{H}_\infty$ controllers, considering the fully actuated configuration.

Note that the nonlinear $\mathcal{H}_\infty$ control via game theory presented the lowest trajectory tracking error $\mathcal{L}_2[\widetilde{x}]$, although the spent energy $E[\tau]$ and the squeeze force $E[h_{oS}]$ are bigger with this controller in comparison with the nonlinear $\mathcal{H}_\infty$ control via quasi-LPV representation.

6.2. *Underactuated configuration*

In this section, joint 1 of the manipulator 1 (on the left of Fig. 1) is passive. In this case, only two components of

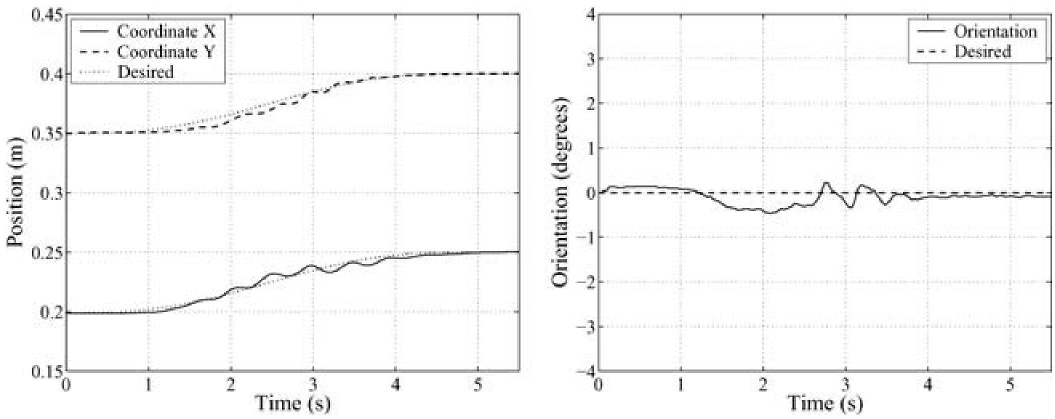

Fig. 3. Fully actuated configuration, control via game theory.

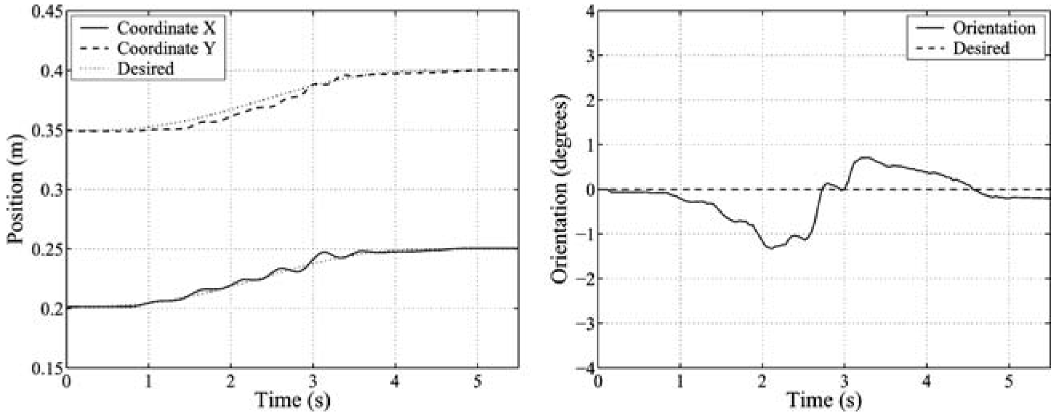

Fig. 4. Underactuated configuration, control via quasi-LPV representation.

the squeeze force can be controlled independently ($n_e = n(m-1) - n_p = 3(2-1) - 1 = 2$).[7] It is defined here that the component of the squeeze force referring to the momentum applied to the object will not be controlled. The desired values for the squeeze force are $\lambda_{Sc}^{d} = [0\ 0]^T$.

The parameter $\rho(\widetilde{x})$, the variation rate bounds, and the basis functions considered to compute $X(\rho)$ are the same used for the fully actuated case. Also, the quasi-LPV system matrices are the same, considering $\widehat{M}(x_o) = \widetilde{M}(x_o)$ and $\widehat{C}(x_o, \dot{x}_o) = \widetilde{C}(x_o, \dot{x}_o)$.

The parameter space was divided considering three points in the set P. The best level of attenuation found was $\gamma = 1.25$. The weighting matrices for the nonlinear $\mathcal{H}_\infty$ control via game theory were also the same as defined for the fully actuated case. The level of attenuation adopted was $\gamma = 4.0$.

The experimental results are shown in Figs. 4 and 5, and the performance indexes in Table III. Note that, in this case, the nonlinear $\mathcal{H}_\infty$ controller via game theory presented the lowest values of trajectory tracking error and of squeeze force. The best value for the spent energy is given by the nonlinear $\mathcal{H}_\infty$ controller via quasi-LPV representation.

Figure 6 presents the squeeze force components when the squeeze force control is applied (continuous line) and when it is not applied (dashed line), for the control via quasi-LPV representation. It can be observed that only two components

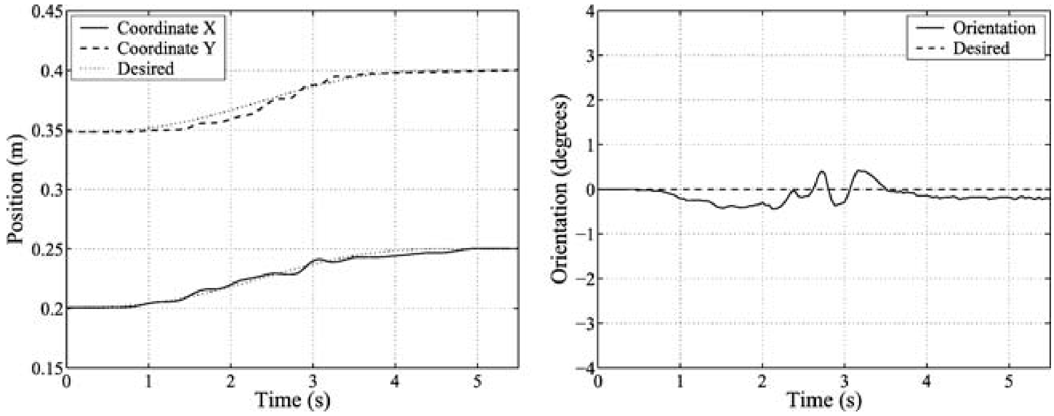

Fig. 5. Underactuated configuration, control via game theory.

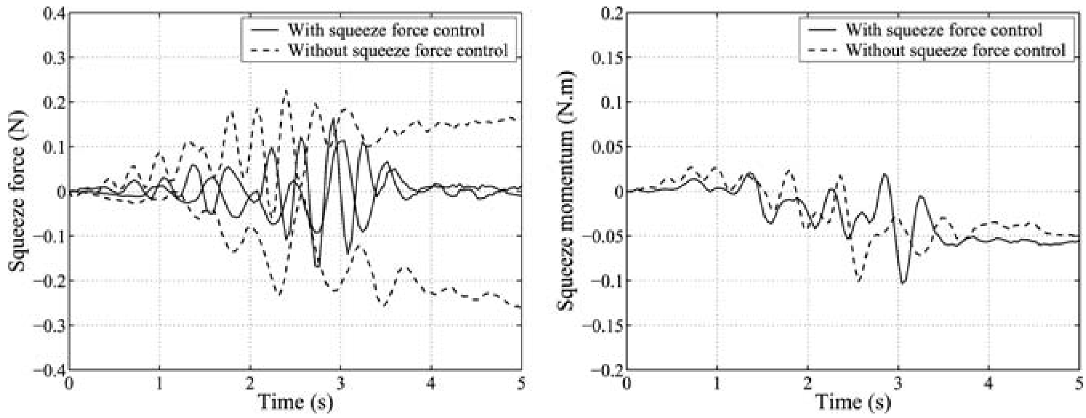

Fig. 6. Squeeze force control.

of the squeeze force related to the linear coordinates are controlled, they are close to the desired values $\lambda^{d}_{Sc} = 0$. The component of the squeeze force related to the momentum is not controlled in both cases, as aforementioned.

For the case where the squeeze force is not controlled, the values of $\mathcal{L}_2[\widetilde{x}]$ and $E[\tau]$ are close to the values of Table III. However, the values of $E[h_{oS}]$, given by 1.6319 N s and 0.9250 N s, for the controllers via quasi-LPV representation and via game theory, respectively, are, in average, three times bigger than the values of $E[h_{oS}]$ for the case where the squeeze force is controlled (see Table III).

The same experiment was also implemented using the hybrid position/force control for underactuated manipulators proposed in ref. 8. The performance indexes are given by: $\mathcal{L}_2[\widetilde{x}] = 0.0128$, $E[\tau] = 1.7781$, and $E[h_{oS}] = 0.5741$. It can be observed that, although the value of $\mathcal{L}_2[\widetilde{x}]$ is lower than that obtained with the controller via quasi-LPV representation, the values of $E[\tau]$ and $E[h_{oS}]$ are approximately 70% and 40%, respectively, bigger than the values obtained with the nonlinear $\mathcal{H}_\infty$ controllers.

7. Conclusion

In this paper, experimental results obtained from the application of nonlinear $\mathcal{H}_\infty$ controls in an actual underactuated cooperative manipulator, subject to parametric uncertainties and external disturbances, are presented. From the computed performance indexes, one can observe that the cooperative system works satisfactorily with one joint not actuated, almost equivalent to the fully actuated case. The results shown confirm that underactuation can be a structural concept to be used by designers, since an appropriate robust control strategy is adopted.

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